Poset topology, Koszul duality and new criteria for shellability

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Rencontres 2025 de topologie algébrique
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Outline

1. Poset topology
2. Shellability and operads [Fresse, Vallette]
3. Parking posets (with L. Randazzo, M. Josuat-Vergès and H. Han)

Poset topology

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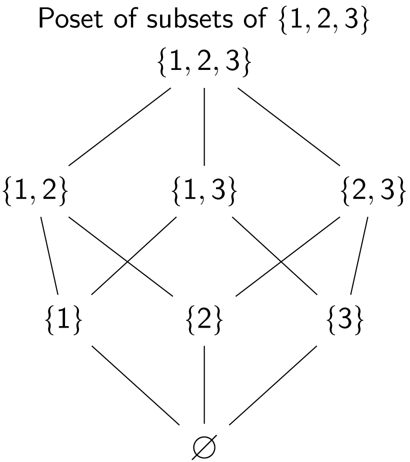
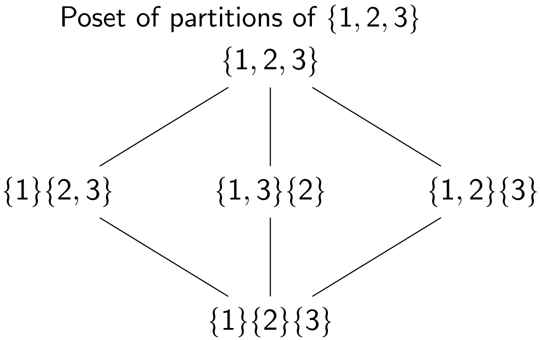
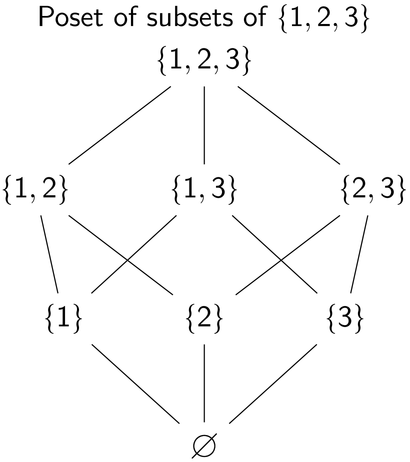
A **poset** is a set S endowed with a partial order. We represent its **Hasse diagram** as a graph whose set of vertices is S and whose edges are covering relations in the poset.

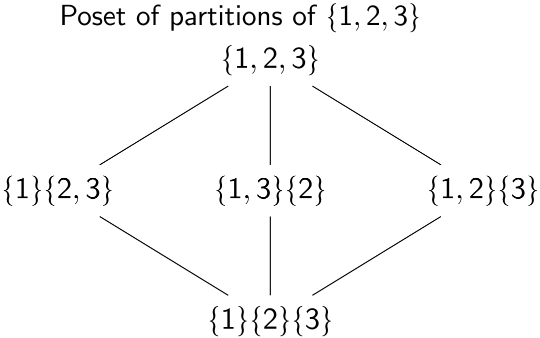




Möbius number of an interval [Rota, 1964]

When the poset has a maximum ($\hat{1}$) and a minimum ($\hat{0}$), it is an **interval** (or bounded poset).
The **Möbius function** is defined recursively by : $\mu(x,x) = 1$ and $\mu(x,y) = -\sum_{x \leqslant z < y} \mu(x,z)$.
The **Möbius number** of the poset is $\mu(P) := \mu(\hat{0}, \hat{1})$.





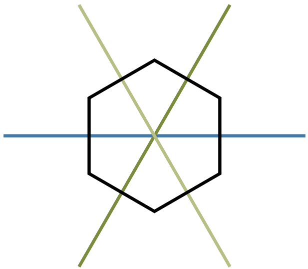
Link with hyperplane arrangement

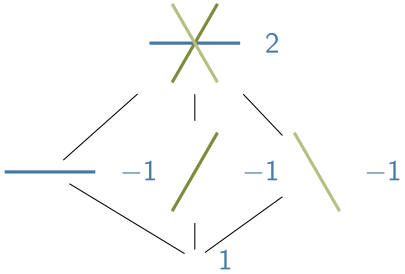
Question

What is the point of computing the Möbius number of a poset ?

Theorem (Zaslavsky's, 1975)

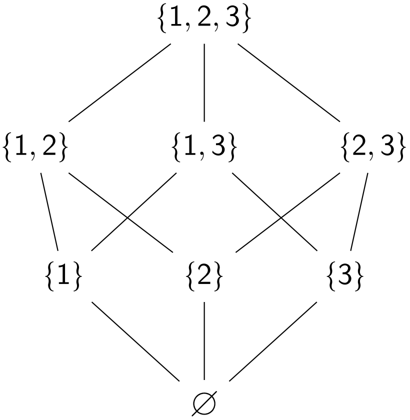
$$\textit{number of k-regions} = \sum_{\substack{I \leqslant J \in L(\mathcal{A}) \\ \dim(I)=k}} |\mu(I,J)|,$$

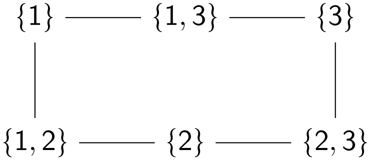




Cohomology of an interval

Möbius number of an interval = Euler characteristic of its order complex (nerve of $P \backslash \{\hat{0}, \hat{1}\}$).





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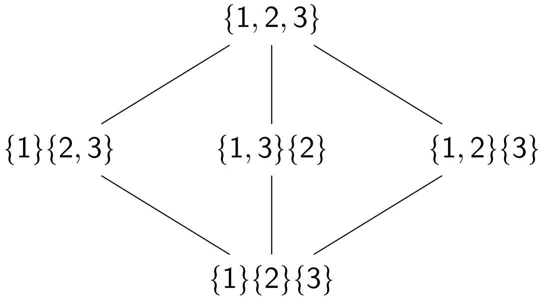


$\{1\}\{2,3\}$ $\{1,3\}\{2\}$ $\{1,2\}\{3\}$

$\rightarrow \tilde{H}^{\bullet}$ reduced cohomology of this order complex

Another definition (in terms of relative cohomology)

We can also consider this alternative cochain complex

$$c^k(P) = \{x_0 < \ldots < x_k \in P | a_0 \in \min(P), a_k \in \max(P)\},$$

endowed with the following differential:

$$d[\gamma] = \sum_{i=1}^{n} (-1)^i \sum_{x_{i-1}<y<x_i} [\cdots < x_{i-1} < y < x_i < \cdots].$$

We denote by $h^\bullet(P)$ its cohomology. It is well defined for any poset P.

Relations between cohomologies (1)

For $n \geqslant 1$, when P is an interval

$$h^n(P) = \tilde{H}^{n-2}(P \backslash \{\hat{0}, \hat{1}\}).$$

When P is not an interval

We can associate to it two other cochain complexes

$$\check{c}^k(P) = \mathbb{K}.\{x_0 < \ldots < x_k | x_0 \in \min(P)\}$$
$$\hat{c}^k(P) = \mathbb{K}.\{x_0 < \ldots < x_k | x_k \in \max(P)\},$$

endowed with:

$$d[\gamma] = \sum_{i=1}^{n} (-1)^i \sum_{x_{i-1}<y<x_i} [\cdots < x_{i-1} < y < x_i < \cdots].$$

The associated cohomology are denoted respectively by $\check{h}(P)$ and $\hat{h}(P)$.

Relations between cohomologies (2)

$$\check{h}^n(P) \simeq \bigoplus_{x \in \min(P)} \widetilde{H}^{n-1}(P_{>x}),$$

$$\hat{h}^n(P) \simeq \bigoplus_{y \in \max(P)} \widetilde{H}^{n-1}(P_{<y}),$$

Definitions

- A poset is **Cohen-Macaulay** if it has the homotopy type of a wedge of spheres of same dimensions. Then it has a unique non trivial reduced cohomology group.
- A poset is **shellable** if there is a linear order on its facet $F_1, \ldots, F_n$ such that all the facets of $\left(\cup_{i=1}^{k-1} \langle F_i \rangle\right) \cap \langle F_k \rangle$ have dimension $\dim F_k - 1$. [Schläfli]

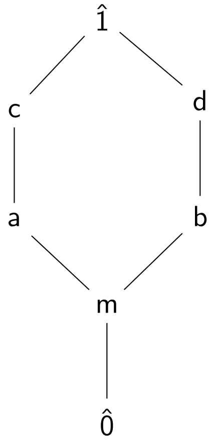
Warning !

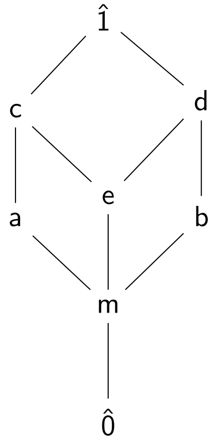
Shellability is not a topological property !

Question

How to determine whether a poset is shellable ?

Two examples





An **edge labelling** of a bounded poset P is a map λ from the edges of the Hasse diagram $E(P)$ (i.e. the covering relations) to $\mathbb{N}$.
To any maximal chain $c = \hat{0} \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_n \lessdot \hat{1}$ can be associated a word $\lambda(c) = \lambda(\hat{0}, x_1)\lambda(x_1, x_2)\ldots\lambda(x_n, \hat{1})$. The chain is **increasing** if $\lambda(\hat{0}, x_1) < \lambda(x_1, x_2) < \ldots < \lambda(x_n, \hat{1})$.

Definition

An edge labelling λ is an EL-labelling if on any interval $[x; y]$

- there exists a unique increasing chain c
- c is minimal in the lexicographic order.

Remark:

There is a weaker notion called CL-shellability in which the pairs $(c, (x_i \lessdot x_{i+1}))$ are labelled, where c is a maximal chain from $\hat{0}$ to x_i.

EL-shellability on an example

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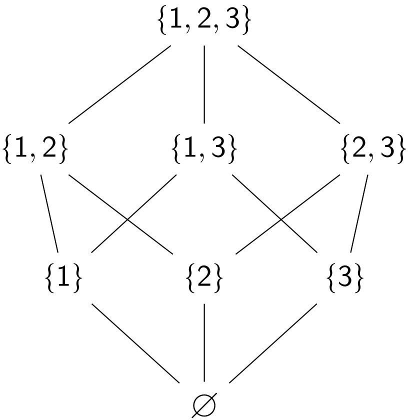
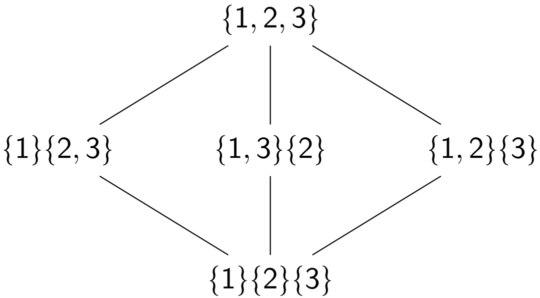
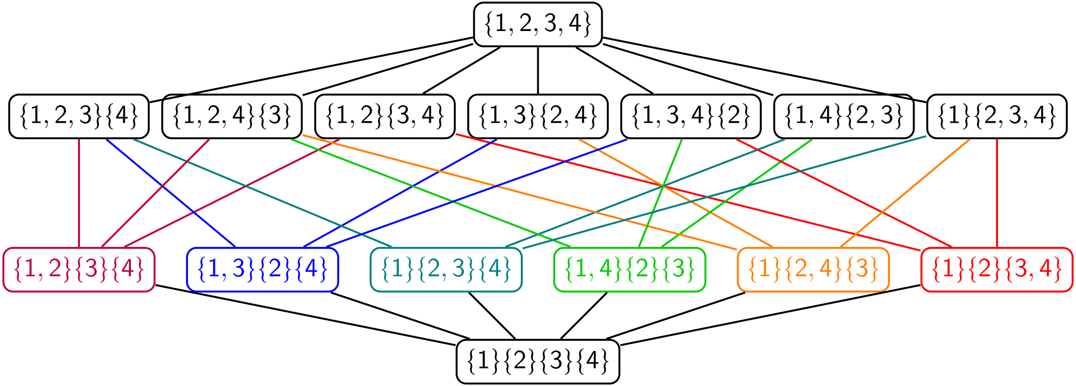




A bigger example

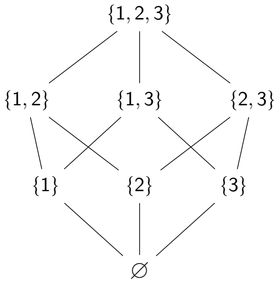


Recursive atom ordering (equivalent to CL-shellability) [Björner-Wachs]

Definition

A bounded poset P admit a recursive atom ordering (RAO) if $\hat{0} \lessdot \hat{1}$ or if there is an ordering $a_1, \ldots, a_t$ of the elements covering $\hat{0}$ (atoms) such that:

- For all $j \in [\![1; t]\!]$, the interval $[a_j; \hat{1}]$ admits a RAO in which the atoms of $[a_j; \hat{1}]$ that belong to $[a_i; \hat{1}]$ for some $i < j$ come first
- For all $i < j$, if $a_i, a_j < y$ then there is a $k < j$ and an atom z of $[a_j; \hat{1}]$ such that $a_k < z \leqslant y$.



Summary

CL-shellability $\Longrightarrow$ EL-shellability $\Longrightarrow$ shellability $\Longrightarrow$ Cohen-Macaulay

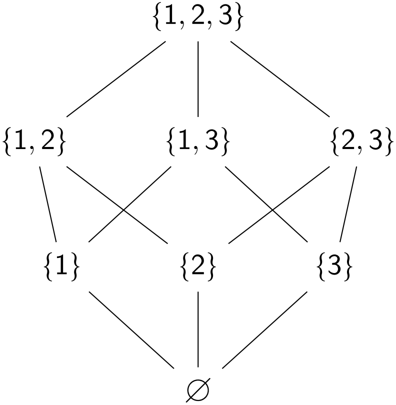
Still other kinds of shellability [Björner, Wachs, Gonzales d'Leon, ...]

Link with polytopes

Convex polytope with orientation vector → poset

Question

On which condition on P is P the 1-skeleton of a convex polytope ?



Also

- Relations poset ↔ algebras
- Intervals in the posets
- Quotient of posets by congruence relations

Shellability and operads [Fresse, Vallette]

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Cohomology of partition posets

Proposition (Hanlon, 81 ; Stanley, 82 ; Joyal 85)

The poset of partitions of a finite set V, $\Pi(V)$, has a unique non trivial cohomology group given by:

$$\mu(\Pi(V)) = (|V| - 1)!$$

Moreover, the action of the symmetric group on this cohomology group is:

$$h^{n-1}(\Pi(V)) = \mathrm{Lie(V)} \otimes_{\mathfrak{S}_V} \mathrm{sgn},$$

where sgn *is the signature representation.*

$\mathrm{Lie}(\{1,2\}) = \mathbb{K}.\{[1;2]\}$ with $[1;2] = -[2;1]$
$\mathrm{Lie}(\{1,2,3\}) = \mathbb{K}.\{[[1;2];3],[[1;3];2]\}$
with $[[1;2];3] + [[2;3];1] + [[3;1];2] = 0$ (Jacobi relations)
$\mathrm{Lie}(\{1,\ldots,n\}) = \mathbb{K}.\{[\ldots[1;\sigma(2)]\sigma(3)]\ldots\sigma(n)],\sigma \in \mathfrak{S}(\{2,\ldots,n\})\}$ [Reutenauer]

$\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\}$

$\{1,5\}\{2\}\{3\}\{4\}\{6\}\{7\}\{8\}\{9\}$

$\{1,5\}\{2\}\{3,8,9\}\{4\}\{6\}\{7\}$

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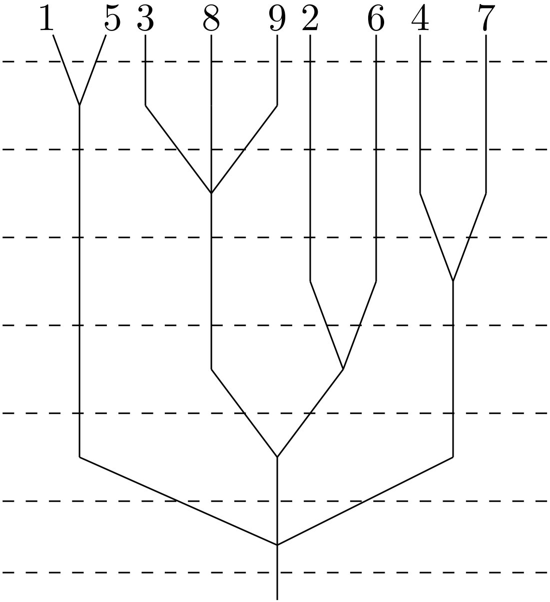


Definition [Joyal, 80s]

A **set species** is a functor

$$\mathbb{F} : \text{Bij} \longrightarrow \text{Set}$$

Bij: category of finite sets and bijections

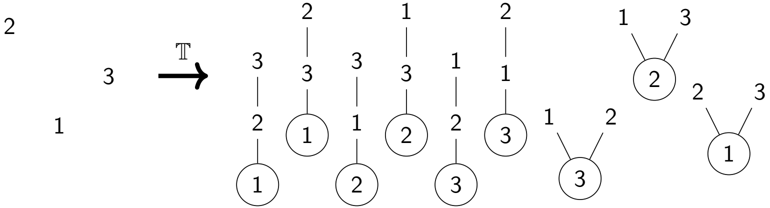
Set: category of sets

A **linear species** is a functor

$$\mathbb{F} : \text{Bij} \longrightarrow \text{Vect} \qquad (+\text{ général})$$

Bij: category of finite sets and bijections

Vect: category of vector spaces



Substitution

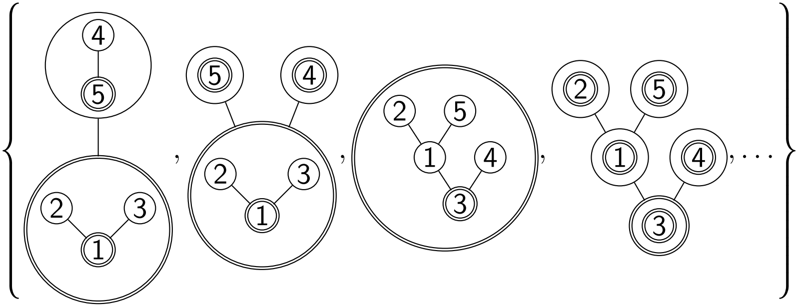
The **substitution** of two species $\mathbb{F}$ and $\mathbb{G}$, with $\mathbb{F}(\varnothing) = \{0\}$ is defined as:

$$(\mathbb{F} \circ \mathbb{G})(E) = \bigoplus_{\pi \in \Pi(E)} \mathbb{F}(\pi) \otimes \bigotimes_{p \in \pi} \mathbb{G}(p)$$

For instance, for $\pi = \{A, B, C\}$ with $A = \{1, 3\}$, $B = \{2\}$ and $C = \{4\}$,

$$\mathbb{L} \circ \mathbb{L} \supseteq (B, A, C) \otimes ((2) \otimes (3, 1) \otimes (4)) = ((2), (3, 1), (4))$$

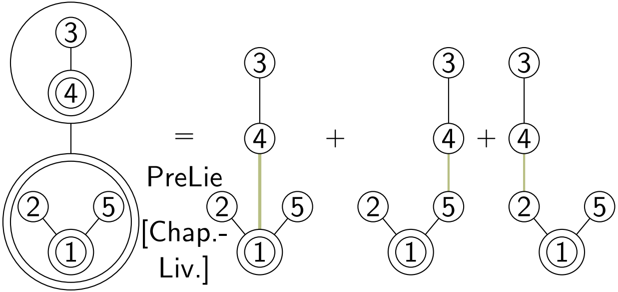
$$\mathbb{T} \circ \mathbb{T}([\![5]\!]) = \mathbb{K}.$$



A (symmetric) (resp. set) operad $\mathcal{O}$ is

- a linear species (resp. set species) $\mathcal{O}$ with an associative composition

$$\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$$



- and a unit $u : \mathbb{X} \to \mathcal{O}$, where $\mathbb{X}$ is the singleton species ($\mathbb{X}(S) = \delta_{|S|=1}\mathbb{C}$).

We consider here connected operads: $\mathcal{P}(\varnothing) = \varnothing$ and $\mathcal{P}(\{*\}) = \{*\}$

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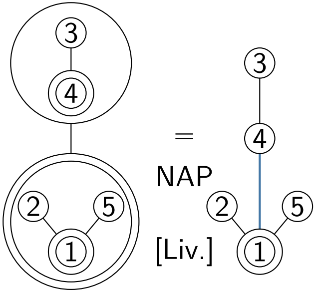

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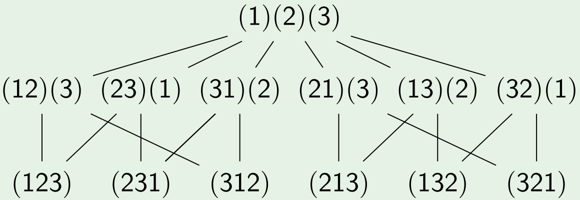
Decorated partition posets [Vallette, 07]

Let $\mathcal{P}$ be a connected set operad. A **$\mathcal{P}$-decorated partition** on a finite set V is an element of $\mathbb{E} \circ \mathcal{P}$.

$$(\alpha, \eta) \leqslant (\beta, \xi) \Leftrightarrow \alpha \leqslant_{\Pi(V)} \beta, \forall A \in \alpha, \exists \nu_A \in \mathcal{P}(\beta_{|A}) \text{ s.t. } \eta_A = \nu_A \circ (\xi_B)_{B \in \beta_{|A}}.$$

Assoc $=$ $\mathbb{L}$-decorated partitions on $\{1, 2, 3\}$



Theorem (Vallette, 07)

$\mathbb{K}\mathcal{P}$ is Koszul iff the associated posets are Cohen-Macaulay. Moreover in that case,

$$h^{|V|-1}(\Pi^{\mathcal{P}}(V)) \simeq s^{n-1} (\mathbb{K}\mathcal{P})^! (V) \otimes_{\mathfrak{S}_V} \mathrm{sgn} =: \Lambda^{-1} (\mathbb{K}\mathcal{P})^! (V).$$

What about PBW operads ? [Bellier-Millès–DO–Hoffbeck, 2021]

- An operad is Koszul if and only if the associated decorated partition posets are Cohen-Macaulay.
- We have seen that there are many properties refining Cohen-Macaulayness.
- Dotsenko-Khoroshkin introduced an algorithmic criterion to determine whether an operad is Koszul : Gröbner/PBW bases.
- When decorated partition posets are CL-shellable, with a compatibility of labellings between subposets, the associated operads admits a PBW bases.
- The converse is not true in general.

ArXiv : 2505.06094

- When a poset P is not bounded, the different definitions give different kinds of cohomologies.
- $\hat{P}$:=minimal interval containing P
- When the poset $\hat{P}$ is CL-shellable, so are the maximal intervals in P.
- The converse is not true in general.

Question

Can we deduce from the topology of maximal intervals in P something on the topology of $\hat{P}$?

Answer for decorated species

By Paul Laubié : When the operad $\mathcal{P}$ is Koszul, it depends on whether $\mathcal{P}$ can be written as $\mathrm{Lie} \circ \mathcal{Q}$ or $\mathcal{Q} \circ \mathrm{Lie}$.
ArXiv : 2510.23547

Parking posets (with L. Randazzo, M. Josuat-Vergès and H. Han)

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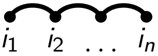
Noncrossing partitions [Kreweras, 1972]

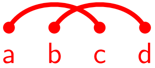
$\{i_1, \ldots, i_n\}$ with $i_1 < \ldots < i_n \rightarrow$ 

Definition (Kreweras, 1972)

A partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of $\{1, \ldots, n\}$ is noncrossing iff

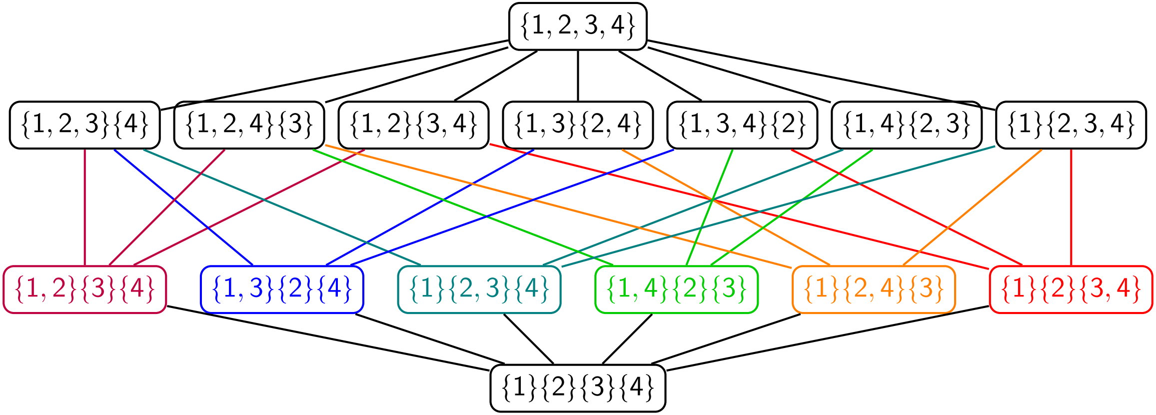
$$\begin{cases} a < b < c < d \\ a, c \in \pi_i \\ b, d \in \pi_j \end{cases} \implies i = j$$

NC_n = set of noncrossing partitions of $\{1, \ldots, n\}$



$\rightarrow$ counted by Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$

Partition posets



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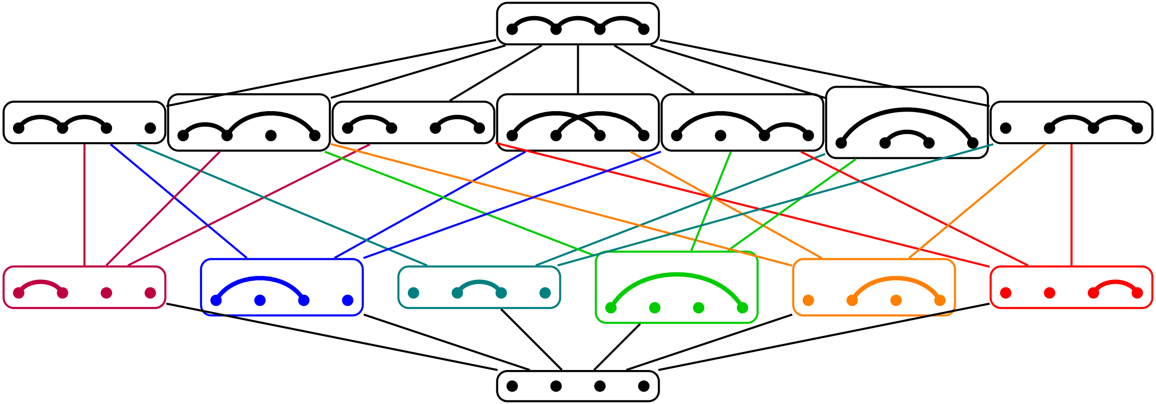
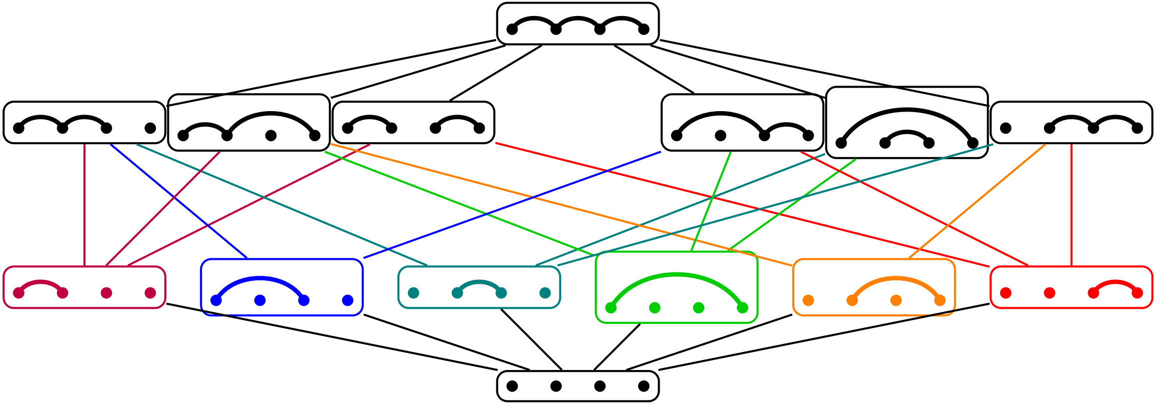


Noncrossing partition posets

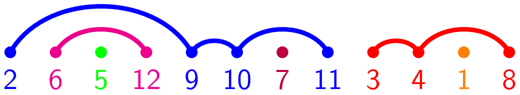


Noncrossing 2-partitions

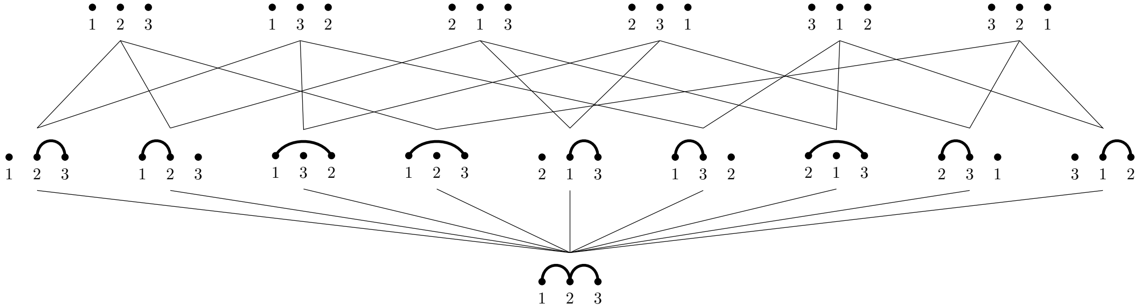
Definition (Edelman, 1980)

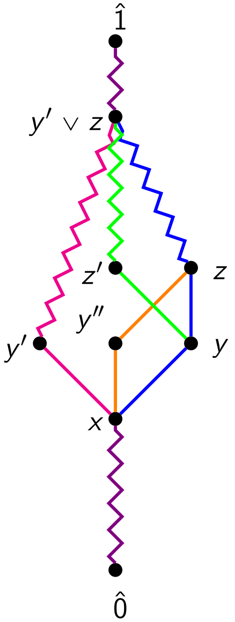
A n.c. 2-partition of size n is a pair $(\pi, \sigma) \in NCP_n \times \mathfrak{S}_n$ s.t.

$$\begin{cases} \{b_1, \dots, b_k\} \in \pi \\ b_1 < b_2 < \dots < b_k \end{cases} \implies \sigma(b_1) < \sigma(b_2) < \dots < \sigma(b_k).$$



Noncrossing 2-partition poset on 3 elements



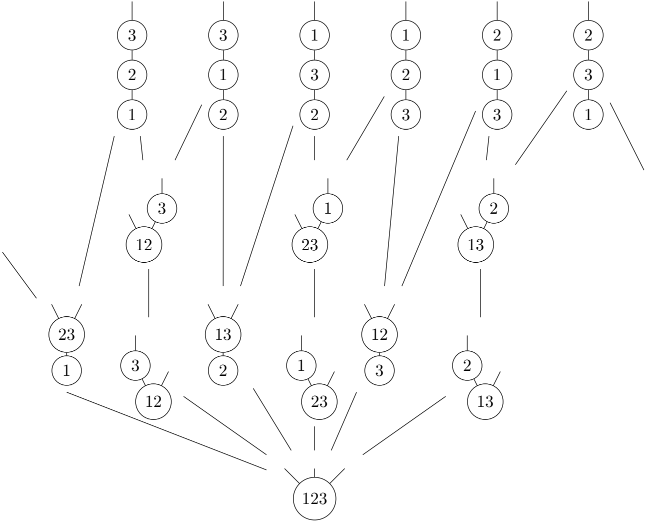


Lemma (D.O., Josuat-Vergès, Randazzo, 22)

Consider a poset P endowed for any element x with an order $\prec_x$ on the atoms of x. If the following condition (C) is satisfied then P is shellable, hence Cohen-Macaulay.

(C) For any $x, y, y', z \in P$ such that $x \lessdot y \lessdot z$, $x \lessdot y'$, and $y' \prec_x y$, then:

- *either there exists $y'' \in P$ such that $x \lessdot y'' \lessdot z$ and $y'' \prec_x y$,*
- *or there exists $z' \in P$ such that $y \lessdot z' \leqslant y' \vee z$ and $z' \prec_y z$.*



Conjecture (DO)

Augmented Tamari-parking posets are homotopic to a sphere.

Proposition (H. Han, 24)

Tamari-parking posets are lattices. They are neither EL-shellable nor CL-shellable.

Question

What is the link between the unique cohomology group of Tamari-parking posets and Associative operad ?

Thank you for your attention !