Orders and shuffles on faces of nestohedra

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Outline

Shuffle products on faces of the nestohedra

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Shuffles

Shuffles (in math)

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^*$ its set of (possibly empty) words (finite sequences of letters).
For any $a, b \in \mathcal{A}$ and $m, n \in \mathcal{A}^*$ the shuffle product is:

$$a.m ⧢ b.n = a.(m ⧢ b.n) + b.(a.m ⧢ n),$$

with $\varepsilon ⧢ m = m ⧢ \varepsilon = m$, where ε is the empty word.

Example :

$ana ⧢ bob =$

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Example :

$ana \sqcup\!\sqcup bob = anabob + anbaob + anboab + anboba + abnaob + abnoab + abnoba + abonab + abonba + abobna$

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Example :

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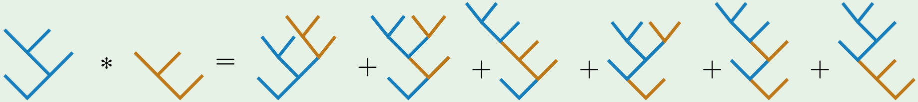
Shuffles of (planar) binary trees [Loday-Ronco 1998]

Denote by $T = \overset{T_l \quad T_r}{\vee}$ a (planar) binary tree.
Loday-Ronco introduced the following product in 1998:

$$T * S = \overset{T_l \quad T_r * S}{\vee} + \overset{T * S_l \quad S_r}{\vee} .$$

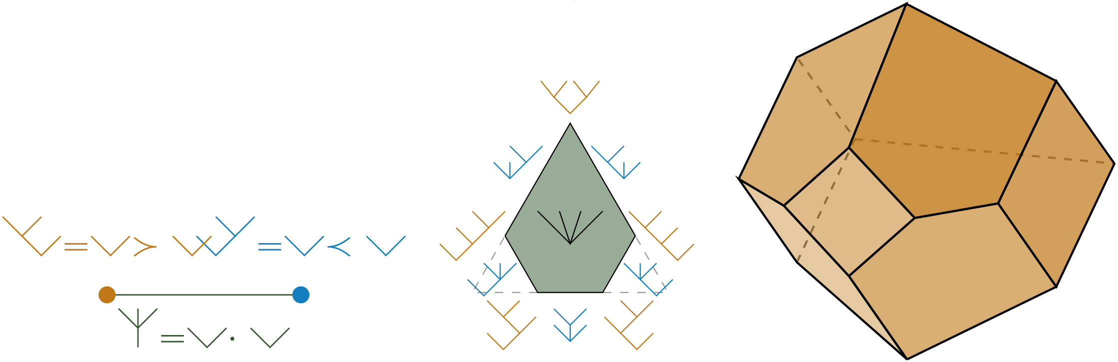
Example



Associahedra

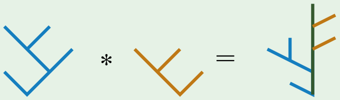
vertices = planar binary trees

k-dimensional face = planar trees/ordered trees on $n-k$ nodes



Shuffles of planar trees [Loday-Ronco, 2004; Chapoton, 2002]

Example



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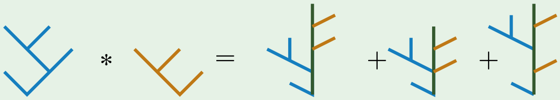

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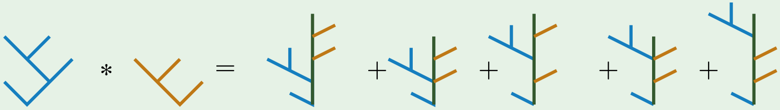

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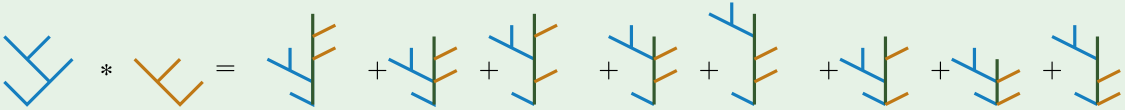

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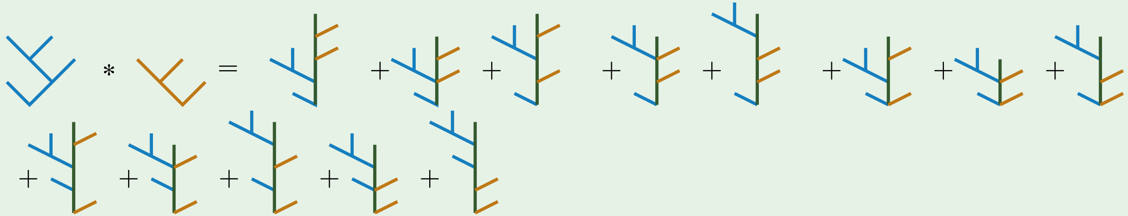

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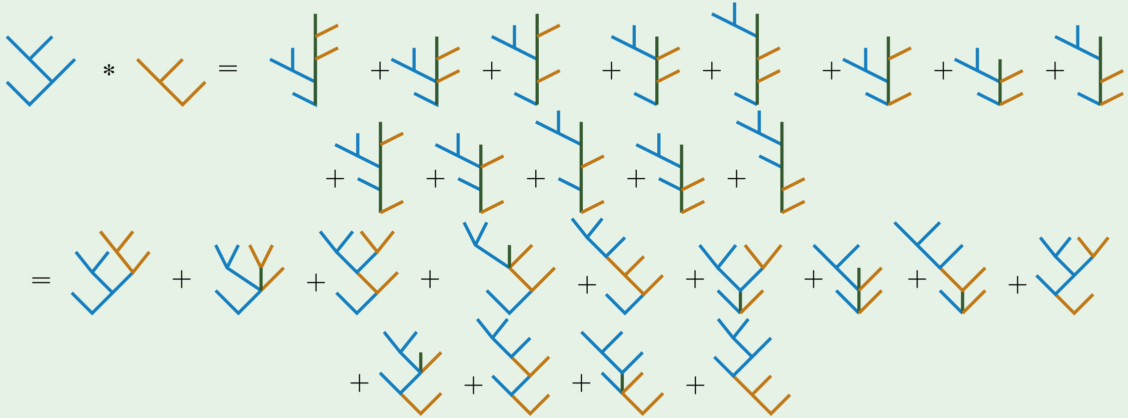

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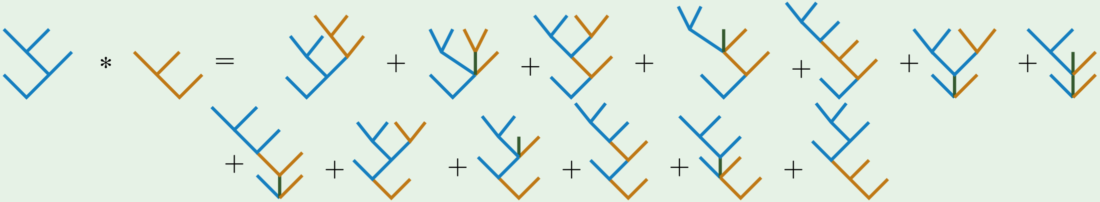

Product $*$ is associative, free associated algebra but generated by infinitely many generators [Loday-Ronco, 1998]

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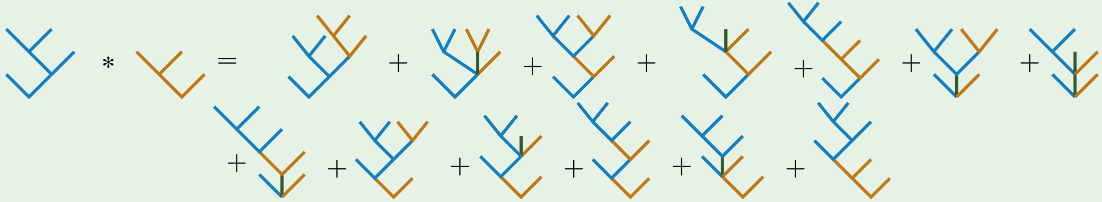
Example



Product $*$ is associative, free associated algebra but generated by infinitely many generators [Loday-Ronco, 1998]

Idea:

Three kinds of trees (looking at the root) : why not splitting in three the product $*$?

Inductive definition of tridendriform products on trees

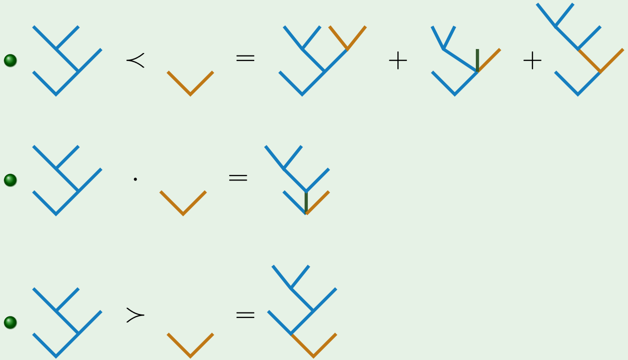
If $T = t_l \vee t_r$ and $S = s_l \vee s_r$,

$$T \prec S = t_l \vee (t_r * S)$$

$$T \cdot S = \vee(t_l,\ t_r * s_l,\ s_r)$$

and $$T \succ S = (T * s_l) \vee s_r$$

Examples :



Tridendriform algebras

Definition (Loday, Ronco, 2004 ; Chapoton 2002)

A tridendriform algebra is a vector space A endowed with products $\prec: A \otimes A \to A$, $\cdot : A \otimes A \to A$ and $\succ: A \otimes A \to A$, such that:

1. $(a \prec b) \prec c = a \prec (b * c)$,
2. $(a * b) \succ c = a \succ (b \succ c)$,
3. $(a \succ b) \prec c = a \succ (b \prec c)$,
4. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
5. $(a \succ b) \cdot c = a \succ (b \cdot c)$,
6. $(a \prec b) \cdot c = a \cdot (b \succ c)$,
7. $(a \cdot b) \prec c = a \cdot (b \prec c)$,

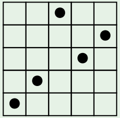
with $* = \prec + \cdot + \succ$

Shuffles of permutations

Let us represent a permutation $\sigma \in \mathfrak{S}_n$ as $\sigma(1)\dots\sigma(n)$. Graphically, we can represent it as a table with a dot at position $(i, \sigma(i))$ for all $1 \leqslant i \leqslant n$.

Example:

$(354) = 12534$ is represented as



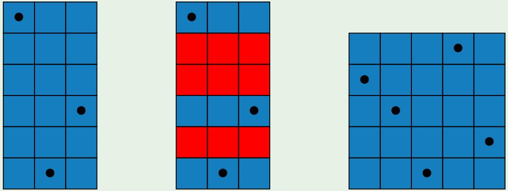
One can define the vertical shuffle of two permutations σ and τ:

$$\sigma \sqcup\!\sqcup_v \tau = \sum_{\substack{\mathrm{std}(s)=\sigma \\ \mathrm{std}(t)=\tau}} s.t,$$

where std is the standardisation, i.e. $\mathrm{std}(t) = \phi \circ t$ with ϕ the unique increasing bijection from $\Im(t)$ to $[\![|\Im(t)|]\!]$.

Example:

$\mathrm{std}(613) = 312$



(Vertical) shuffle of permutations

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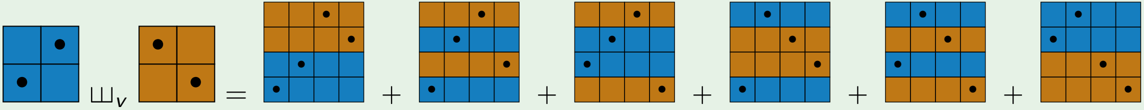
Example:

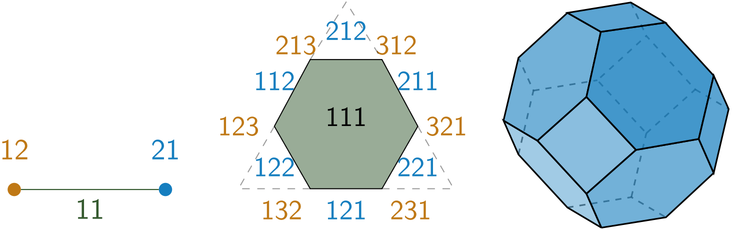
$12 \sqcup\!\sqcup_v 21 = 1243 + 1342 + 2341 + 1432 + 2431 + 3421$



Permutohedra [Schoute 1911]

vertices=permutation of $[\![1;n]\!]$

k-dimensional face = a surjection from $[\![1;n]\!]$ to $[\![1;k]\!]$ (also know as packed word)



Shuffles of packed words/surjections [Perm : Chapoton 2000, WQSym: Hivert-Novelli-Thibon ∼2000, NQSym* : Bergeron-Zabrocki, 2005]

Definition

A packed word is a word w on $\mathbb{N}$ such that the set of letters in w is an interval $[\![1;k]\!]$. Equivalently, it is a surjection.

$$u \sqcup\!\sqcup v = \sum_{\substack{\text{pack}(\alpha)=u \\ \text{pack}(\beta)=v}} \alpha\beta,$$

Example :

$$11 \sqcup\!\sqcup 221 = 22221 + 33221 + 22331 + 11221 + 11332$$

Shuffles of packed words/surjections [Perm : Chapoton 2000, WQSym: Hivert-Novelli-Thibon ∼2000, NQSym* : Bergeron-Zabrocki, 2005]

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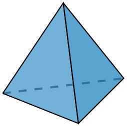
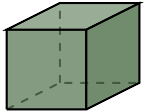
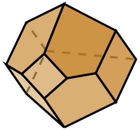
In fact, more than a shuffle product : tridendriform products (WQSym free tridendriform algebra on infinitely many generators [Vong, Burgunder-Curien-Ronco, 2015])



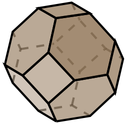
Simplices



Hypercubes



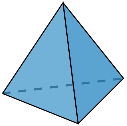
Associahedra



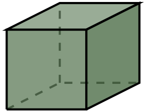
Permutohedra

These two polytopes are instances of nestohedra / hypergraph polytopes

Could we define a shuffle product on their faces ?



Simplices



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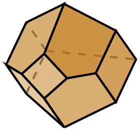

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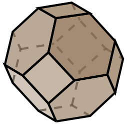

Permutohedra

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Could we define a shuffle product on their faces ?

Answer

Yes, we can !

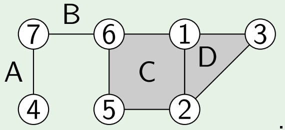
Hypergraphs

Definition

A hypergraph (on vertex set V) is a pair (V, E) where:

- V is a finite set, (the vertex set)
- E is a set of sets of size at least 2, $E \subset \mathcal{P}(V)$.

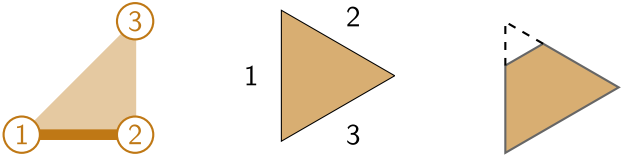
Example of an hypergraph on $[1;7]$



Warning:

All the hypergraphs considered in this talk are connected !

Hypergraph polytope [Došen, Petrić] (=nestohedra [Postnikov])



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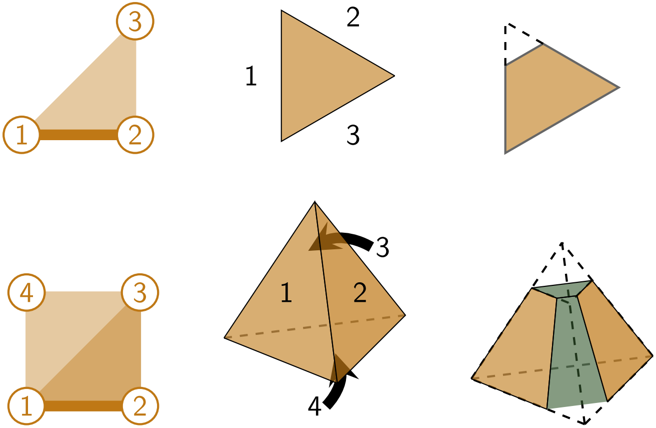


Constructs (aka. tubings, aka. spines) [Postnikov; Curien-Ivanovic-Obradović]

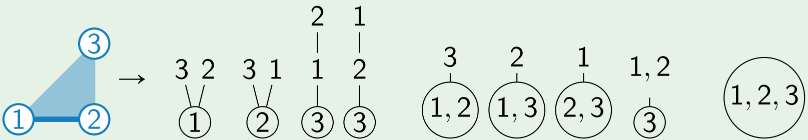
Constructs

A construct c of a hypergraph H is a rooted tree whose vertices form a partition of $V(H)$ and defined inductively by:

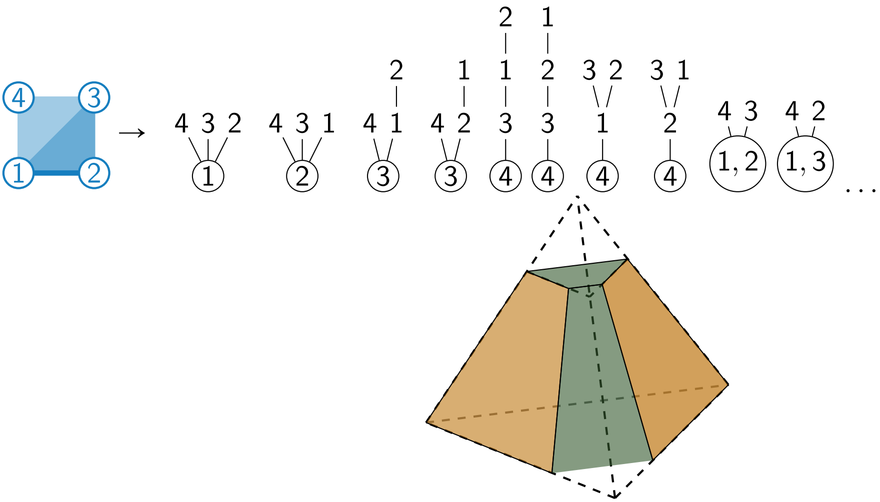
- c has only one node labelled by $V(H)$,
- or the root of c is $E \subseteq V(H)$ and each of its children is a construct of a connected component of $H - E$.

The set of constructs of a given hypergraph labels faces of the associated polytope.

First example:



Second example geometrically



Heuristics for a tridendriform structure

Let $\mathbf{H}^{\mathcal{X}}$ be a family of hypergraph polytopes, indexed by some finite sets $\mathcal{X}$ (sets of vertices of the associated hypergraphs).
For $S = A(S_1, \ldots, S_m)$ and $T = B(T_1, \ldots, T_n)$ two constructs of $\mathbf{H}^{\mathcal{X}}$ and $\mathbf{H}^{\mathcal{Y}}$ respectively ($\mathcal{X}, \mathcal{Y}$ disjoint), we would like to define the following operations

- $S \prec T$ as a sum of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** A**,**
- $S \succ T$ as a sum of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** B**,**
- $S \cdot T$ as a sum of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** $A \cup B$**.**

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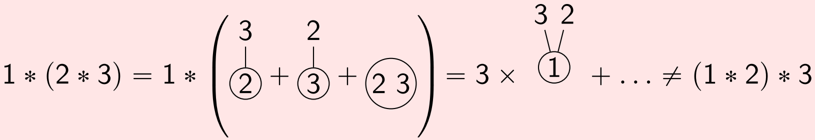
First problem in hypercubes



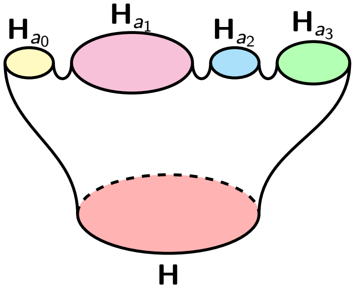
Universe and preteam

The considered hypergraphs belong to a set of hypergraphs $\mathfrak{U}$, called universe.

A preteam (=domain of definition) is a pair $\tau = (\{\mathbf{H}_a | a \in A\}, \mathbf{H})$ where

- $\{\mathbf{H}_a | a \in A, \mathbf{H}_a \in \mathfrak{U}\}$ is a set of pairwise disjoint hypergraphs, called participating hypergraphs
- $\mathbf{H} \in \mathfrak{U}$ is a hypergraph such that $H = \bigcup_{a \in A} H_a$, called supporting hypergraph.

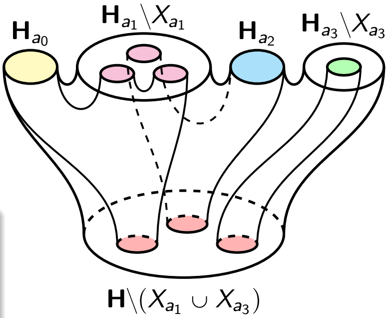
$$* : \prod_{a \in A} \mathcal{C}(\mathbf{H}_a) \to \mathcal{C}(\mathbf{H})$$



Strict and quasi-strict teams [Curien–D.O.–Obradovic, 25]

A preteam is a (resp. quasi-strict) strict team
if the connected components obtained by deleting a subset X_a to every hypergraph $\mathbf{H}_a$ are

- in $\mathfrak{U}$
- and included in the connected components of $\mathbf{H}\backslash\left(\bigcup_{a\in A} X_a\right)$ (resp. or totally disconnected).



$(X_{a_0} = X_{a_2} = \varnothing)$

Examples:

- Strict teams : Associahedra, Permutohedra, Restrictohedra, ...
- Quasi-strict teams : Simplices, Hypercubes, Erosohedra, ...

Shuffle product

Considering a team E and denoting by δ a tuple of constructs of the team's participating hypergraphs, we inductively associate to δ a sum of constructs of the supporting hypergraph:

$$*(\delta) = \sum_{\emptyset \subset B \subseteq A} q^{|B|-1} *_B (\delta),$$

where

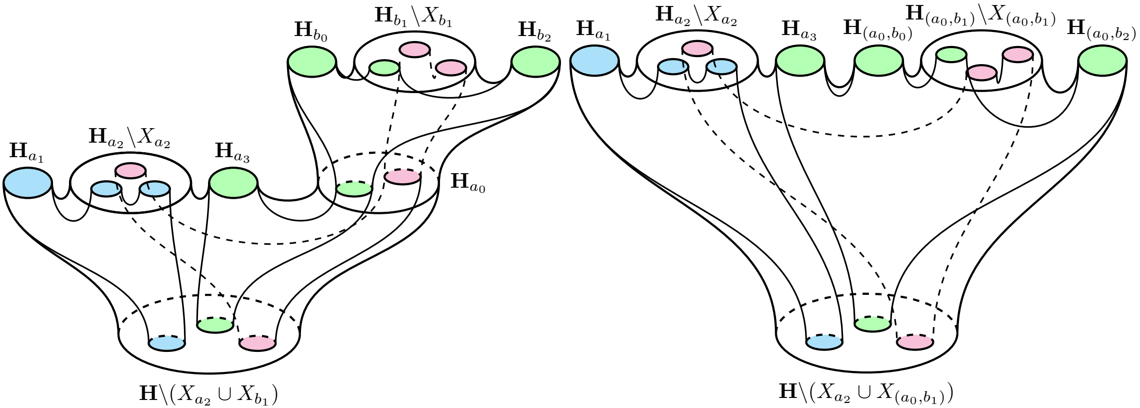
$$*_B(\delta) = (\bigcup_{b \in B} X_b)(*(\delta_1^B), \ldots, *(\delta_{n_B}^B)).$$

Proposition (Curien-D.O.-Obradovic, 25)

$$*_B(\delta) \quad = \quad \sum_{U:\mathbf{H},\ root(U)=X_B \text{ and } \forall a \in A, U_{|\mathbf{H}_a}=C_a} q^{\mu^\tau(U)-|B|+1} U.$$

Associative clan

A set of (resp. quasi-strict) strict team with "good" closure properties is called strict clan (each connected component obtained from the supporting hypergraph is itself a supporting hypergraph of a team).



Associativity of $*$

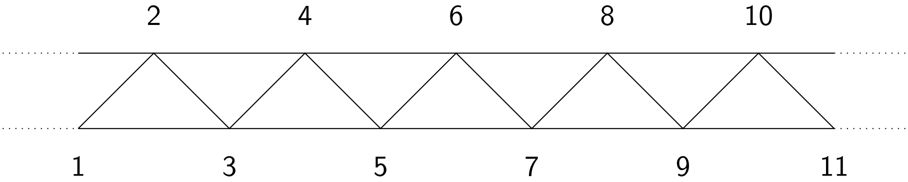
Theorem (Curien-D.O.-Obradović, 25)

Consider an associative clan $\mathcal{C}$. The product $*$ is associative if

- *$\mathcal{C}$ is strict,*
- *or $\mathcal{C}$ is quasi-strict and $q = -1$.*

- Strict clans: Associahedra, Permutohedra, Restrictohedra, ...
- quasi-strict clans: Simplices, Hypercubes, ...

Friezohedra



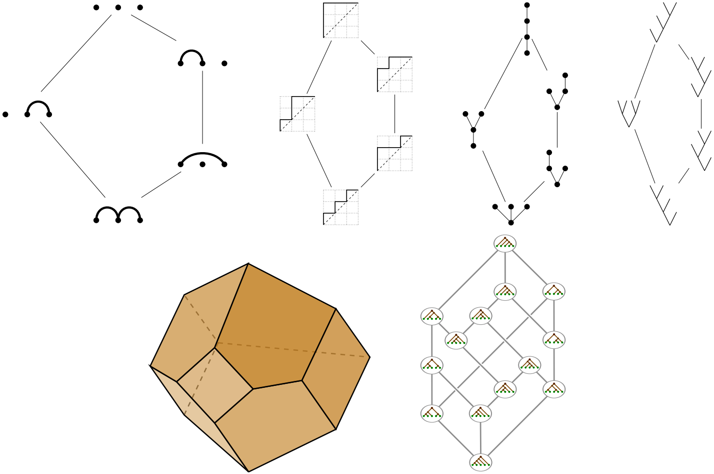
- strict associative clan
- closed by translation
- between associahedra and permutohedra

Order on faces of nestohedra

Outline

1. Shuffle products on faces of the nestohedra
2. Order on faces of nestohedra
3. Link between the two algebraic structures

Tamari order [Tamari 62] (extended to planar trees [Palacios-Ronco, 2002])



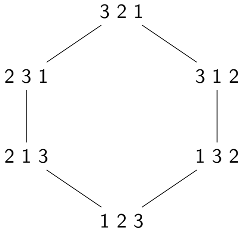
Weak Bruhat order [Verma 1968] (extended to packed words [Krob-Latapy-Novelli-Phan-Schwer, 2001])

1234

Covering relations :

$\dots ab \dots \lhd \dots ba \dots$

with $a < b$



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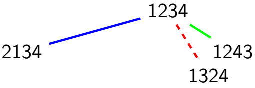

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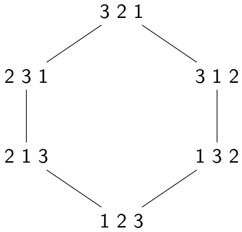

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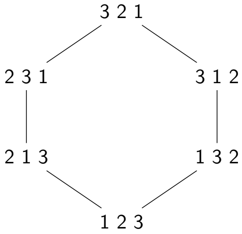
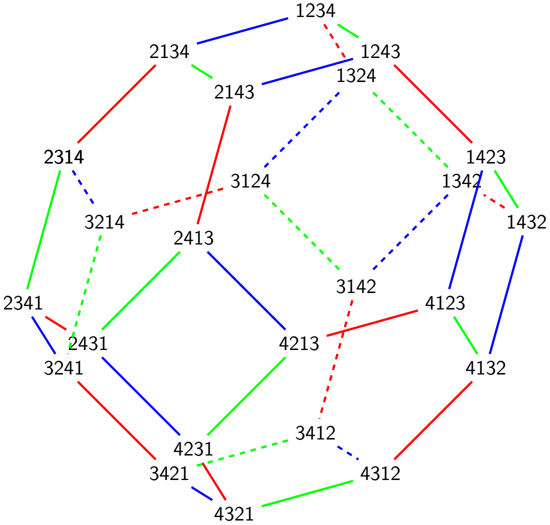
$\ldots ab \ldots \lhd \ldots ba \ldots$

with $a < b$



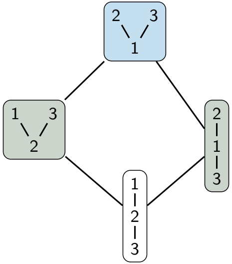


Barnard-McConville's Flip order on vertices of graph associahedra

Let us consider two constructions S and T (with vertices of size 1). Barnard and McConvile define :

$$S \leqslant_{BM} T$$

if and only if
there exists $y < x$ such that x is a parent of y in S and T is obtained from S by exchanging x and y



Order on faces of nestohedra : generalised flip order [Curien–Laplante-Anfossi, Curien–D.O.–Obradovic]

- Hypergraphs on $\mathbb{Z}$
- For $X_1, X_2 \subseteq \mathbb{Z}$, we write $X_1 < X_2$ if $\max(X_1) < \min(X_2)$.
- Every considered set of hypergraphs (preteam, decomposition) is ordered

Definition

S and T two constructs of $\mathbf{H}$. $S \lessdot T$ if and only if there exists

- X parent of Y in S such that $\min(X) > \max(Y)$ and T obtained from S by merging X and Y, (contraction)
- or Y parent of X in T such that $\min(X) > \max(Y)$, and S obtained from T by merging X and Y. (split)

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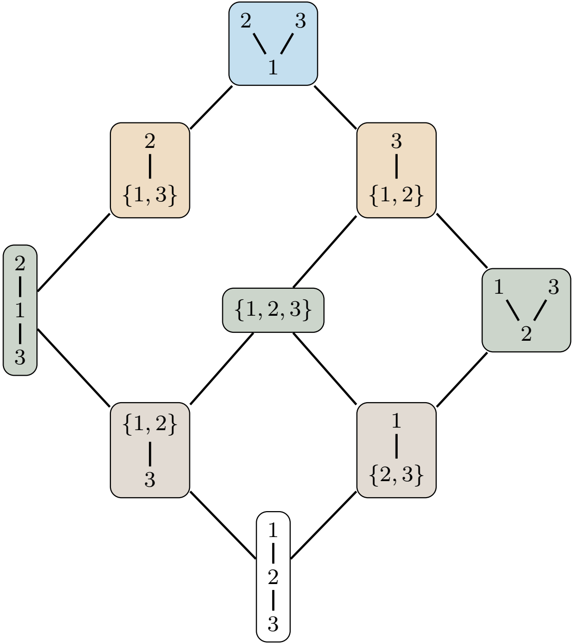
Proposition

The restriction of this order to vertices is Barnard-McConville's order.



2 3 1
2 {1, 3}
3 {1, 2}
2 1 3
{1, 2, 3}
1 3 2
{1, 2} 3
1 {2, 3}
1 2 3

Link between the two algebraic structures

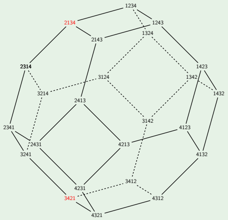
Outline

Link between shuffle products and orders

Proposition (Loday-Ronco 2002, Palacios-Ronco 2006)

The shuffle product of two planar binary trees (resp. planar trees, resp. permutations, resp. pacjed words) can be expressed as a sum of elements in an interval of the Tamari order (resp. generalised Tamari order, resp. weak Bruhat order, resp. geenralised weak Bruhat order) whose bounds are given by some (duplicial) operations on these elements.

Example:



$$21 \sqcup\!\sqcup 12 = \sum_{2134 \leqslant p \leqslant 3421} p$$

$$= 2134 + 2314 + 2341 + 3214 + 3241 + 3421$$

General case

Proposition (Ronco 2012)

For any strict ordered associative clan of graph associahedra, the shuffle product is the sum of elements in an interval in the generalised Tamari order.

Proposition (Curien–D.O.–Obradovic)

For any strict ordered associative clan of nestohedra, the polydendriform products are given by:

$$*_B^{\leqslant}(\delta) = \sum_{U:\mathbf{H} \text{ and } \diagup^B(\delta) \leqslant U \leqslant \diagdown_B(\delta)} q^{\mu^\tau(U)-|B|+1}\, U,$$

where $\diagup^B(\delta)$ *and* $\diagdown_B(\delta)$ *are defined as follows:*

$$\begin{aligned} \diagup^B(\delta) &= (\sqcup_{b\in B} X_b)\left(\diagup(\delta_1^B),\ldots,\diagup(\delta_{n_B}^B)\right) \\ \diagdown_B(\delta) &= (\sqcup_{b\in B} X_b)\left(\diagdown(\delta_1^B),\ldots,\diagdown(\delta_{n_B}^B)\right). \end{aligned}$$

Hopf algebra of packed words/surjections [WQSym: Hivert-Novelli-Thibon ≡2000, NQSym* : Bergeron-Zabrocki, 2005, Perm : Chapoton 2000]

Given a packed word u, define:

$$\Delta(u) = \sum_{k=0}^{n} u_{|[\![1;k]\!]} \otimes u_{|[\![k+1;l]\!]}$$

Example:

$\Delta(11332) = \varepsilon \otimes 11332 + 11 \otimes 221 + 112 \otimes 11 + 11332 \otimes \varepsilon$

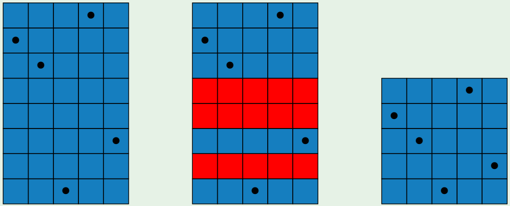
Malvenuto-Reutenauer Hopf algebra [$\mathfrak{S}Sym$: Malvenuto-Reutenauer 1995, "FQSym" : Duchamp-Hivert-Thibon, 2002]

$$\Delta(\sigma) = \sum_{p=0}^{n} \mathrm{std}\,(\sigma(1)\ldots\sigma(p)) \otimes \mathrm{std}\,(\sigma(p+1)\ldots\sigma(n)),$$

where std is the standardisation, i.e. $\mathrm{std}(t) = \phi \circ t$ with ϕ the unique increasing bijection from $\Im(t)$ to $[\![|\Im(t)|]\!]$.

Example:

$\mathrm{std}(76183) = 43152$



Example:

$\Delta(132) = \varepsilon\otimes 132 + 1\otimes 21 + 12\otimes 1 + 132\otimes\varepsilon.$

Proposition

$\otimes_{n\geqslant 0}\mathbb{C}\mathfrak{S}_n$ endowed with the product $\sqcup\!\sqcup_h$ and the coproduct Δ is a Hopf algebra, i.e. Δ is an algebra morphism.

Work in progress

- Link with Hollweg-Dermenjian-Pilaud's weak facial order
- New semi-strict condition to include cyclohedra
- How to include hypercubes for $q = 1$?
- Coproducts ?

Question: Would the following coproduct work ?

$$\Delta(S) = \sum_{\text{c cut}} R_c(S) \otimes *(F_c(S)) + 1 \otimes S$$

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Thank you very much for your attention !