From partition posets to operadic poset species

Bérénice Delcroix-Oger
joint work with Clément Dupont (IMAG)
and Guillaume Laplante-Anfossi (Syddansk University), Kurt Stoeckl (Melbourne University) et Vincent Pilaud (Universitat de Barcelona)

Nantes, Thursday, March 13th 2025

Goal for today

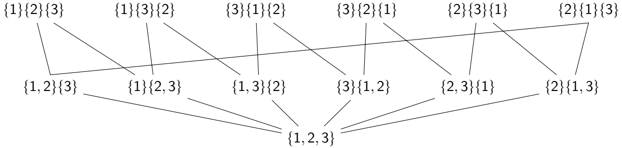
Present an overview of two recent works based on partition posets:

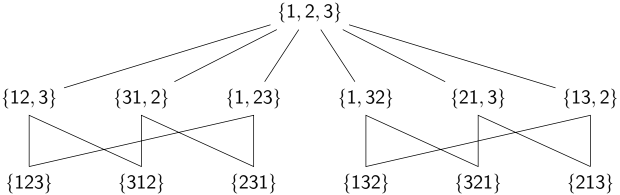
- "Cellular diagonals of permutahedra" joint with G. Laplante-Anfossi (Univ. Syddansk), Kurt Stoeckl (Univ. Melbourne) and Vincent Pilaud (Univ. Barcelone), ArXiv : 2308.12119
- "Lie-operads from poset cohomology" joint with C. Dupont (IMAG), soon on ArXiv

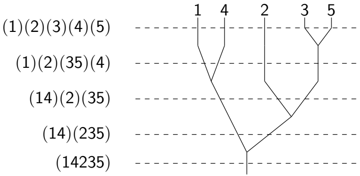
Outline

1. Decorated partition posets
2. Regions of ℓ copies of the braid arrangement and cohomology of decorated partition posets
3. Operadic poset species and its applications

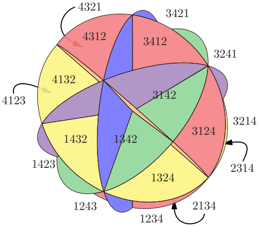
Spoiler of part 1







Spoiler of part 2



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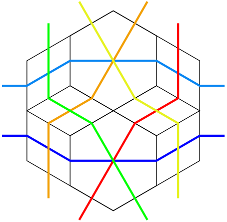

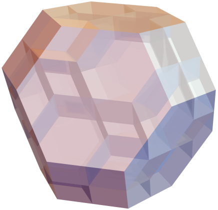

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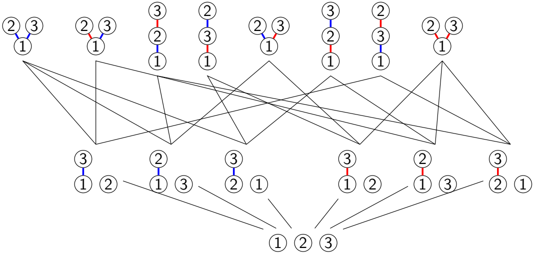
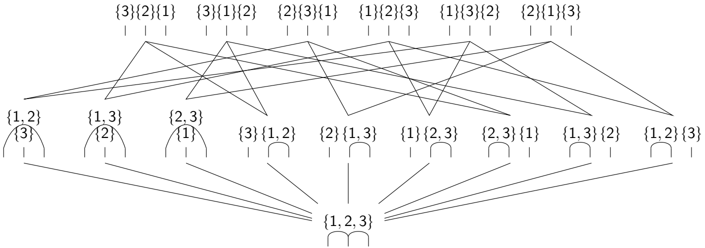
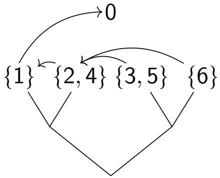
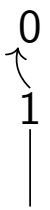
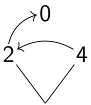


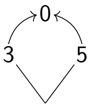
Spoiler of part 3

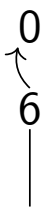


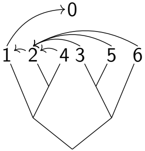
$m_\pi =$ 

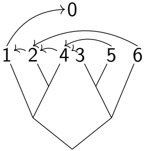
, $m_{\{1\}} =$ 

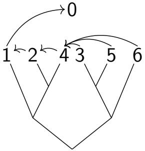
, $m_{\{2,4\}} =$ 

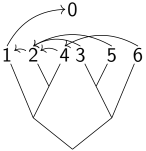
, $m_{\{3,5\}} =$ 

, $m_{\{6\}} =$ 

$\gamma(t(m_\pi), (t(m_p))_{p\in\pi}) =$ 

$+$ 

$+$ 

$+$ 

Partition posets and decorated partition posets

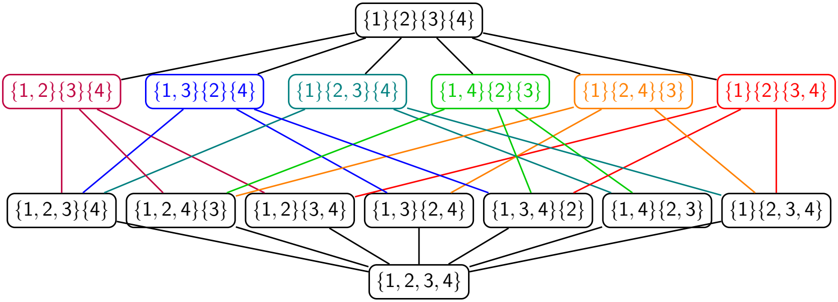
Posets of (set) partitions $\Pi(V)$

Partitions of a set V :

$$\{V_1, \ldots, V_k\} \models V \Leftrightarrow V = \bigsqcup_{i=1}^{k} V_i, V_i \cap V_j = \varnothing \text{ for } i \neq j$$

Partial order on set partitions of a set V:

$$\{V_1, \ldots, V_k\} \leqslant \{V'_1, \ldots, V'_p\} \Leftrightarrow \forall i \in \{1, p\}, \exists j \in \{1, k\} \text{ s.t. } V'_i \subseteq V_j$$



Poset (relative) cohomology

To any poset P can be associated a cochain complex

$$c^k(P) = \{x_0 < \ldots < x_k \in P | a_0 \in \min(P), a_k \in \max(P)\},$$

with the following differential:

$$d[\gamma] = \sum_{i=1}^{n} (-1)^i \sum_{x_{i-1} < y < x_i} [x_0 < x_1 < \cdots < x_{i-1} < y < x_i < \cdots < x_{n-1} < x_n].$$

We denote by $h^\bullet$ the cohomology of $c^\bullet(P)$.

Remark:

When P is bounded, $h^n(P) = \tilde{H}^{n-2}(P \backslash \{\hat{0}, \hat{1}\})$.

Other cohomologies

By considering

$$\check{c}^{\,k}(P) = \mathbb{K}.\{x_0 < \ldots < x_k | x_0 \in \min(P)\}$$

and

$$\hat{c}^{\,k}(P) = \mathbb{K}.\{x_0 < \ldots < x_k | x_k \in \max(P)\},$$

we get $\check{h}(P)$ and $\widehat{h}(P)$.
For $n \geqslant 1$, we have

$$\check{h}^n(P) \simeq \bigoplus_{x \in \min(P)} \widetilde{H}^{n-1}(P_{>x}),$$

$$\widehat{h}^n(P) \simeq \bigoplus_{y \in \max(P)} \widetilde{H}^{n-1}(P_{<y}),$$

Cohomology of the partition poset

Proposition (Hanlon, 81 ; Stanley, 82 ; Joyal 85)

The partition poset $\Pi(V)$ has a unique (co)homology group whose dimension is given by:

$$\mu(\Pi(V)) = (|V| - 1)!$$

Moreover, the action of the symmetric group on this homology group is:

$$h^{n-1}(\Pi(V)) = \mathrm{Lie(V)} \otimes_{\mathfrak{S}_V} \mathrm{sgn},$$

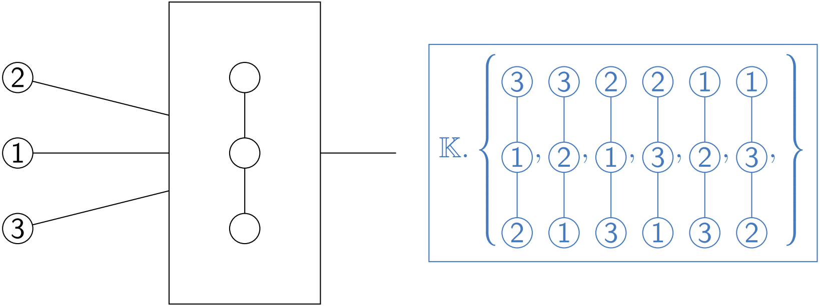
where sgn *is the signature representation.*

$\mathrm{Lie}(\{1,2\}) = \mathbb{K}.\{[1;2]\}$ with $[1;2] = -[2;1]$
$\mathrm{Lie}(\{1,2,3\}) = \mathbb{K}.\{[[1;2];3],[[1;3];2]\}$
with $[[1;2];3] + [[2;3];1] + [[3;1];2] = 0$ (Jacobi relation)
$\mathrm{Lie}(\{1,\ldots,n\}) = \mathbb{K}.\{[\ldots[1;\sigma(2)]\sigma(3)]\ldots\sigma(n)], \sigma \in \mathfrak{S}(\{2,\ldots,n\})\}$
[Reutenauer]

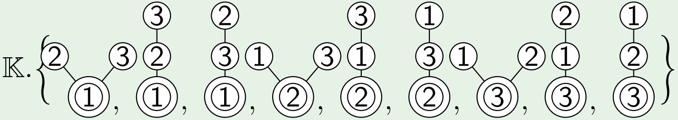
What are species?

Definition (Joyal, 80s)

A set species F is a functor from Bij to Set.
A linear species L is a functor from Bij to $\mathbb{K}$-Mod.



Examples of species

- $\mathbb{K}.\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists Assoc on $\{1,2,3\}$)
- $\mathbb{K}.\{\{1,2,3\}\}$ (Species of non-empty sets Comm)
- $\mathbb{K}.\{\{1\},\{2\},\{3\}\}$ (Species of pointed sets Perm)
-  (Species of Cayley trees $\mathbb{T}$)
- $\mathbb{K}.\{[[1,2],3],[[1,3],2]\}$ (Species of Lie brackets Lie)

These modules are the image by species of the set $\{1,2,3\}$. All but the last one come from linearisations of set species.

Substitution of species

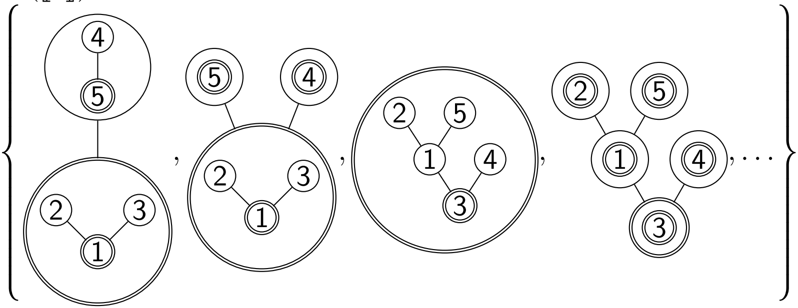
Proposition

Let F and G be two species. Let us define:

$$(F \circ G)(S) = \bigoplus_{\pi \in \Pi(S)} F(\pi) \otimes \bigotimes_{J \in \pi} G(J),$$

where $\Pi(S)$ runs on the set of partitions of S.

$\mathbb{T} \circ \mathbb{T}(\llbracket 5 \rrbracket) =$

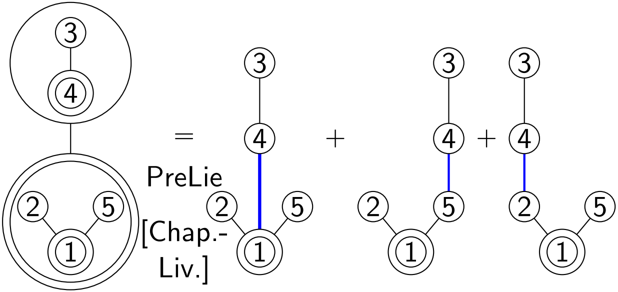
$\mathbb{K}.$ 

Operads

A (symmetric) operad (resp. set operad) $\mathcal{O}$ is

- a linear species (resp. set species) $\mathcal{O}$ with an associative composition

$$\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$$



- and a unit $i : I \to \mathcal{O}$, where I is the singleton species ($I(S) = \delta_{|S|=1}\mathbb{C}$).
- To each kind of algebra is associated an operad.

Operads here will be connected : $\mathcal{P}(\varnothing) = \varnothing$ and $\mathcal{P}(\{*\}) = \{*\}$

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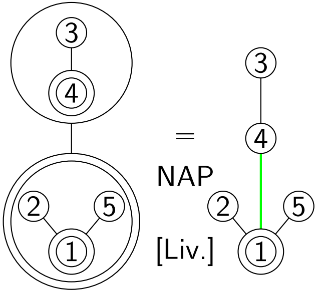

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Levelled (co)bar construction [Fresse, 02]

$\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\}$

$\{1,5\}\{2\}\{3\}\{4\}\{6\}\{7\}\{8\}\{9\}$

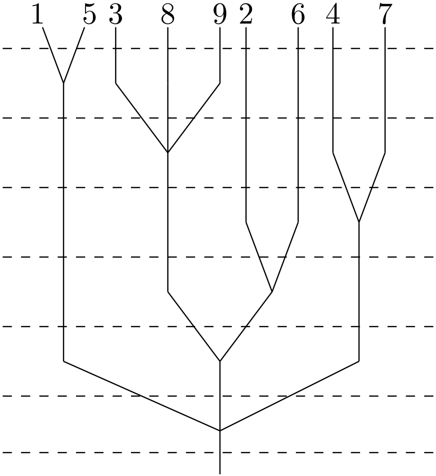
$\{1,5\}\{2\}\{3,8,9\}\{4\}\{6\}\{7\}$

$\{1,5\}\{2\}\{3,8,9\}\{4,7\}\{6\}$

$\{1,5\}\{2,6\}\{3,8,9\}\{4,7\}$

$\{1,5\}\{2,3,6,8,9\}\{4,7\}$

$\{1,2,3,4,5,6,7,8,9\}$



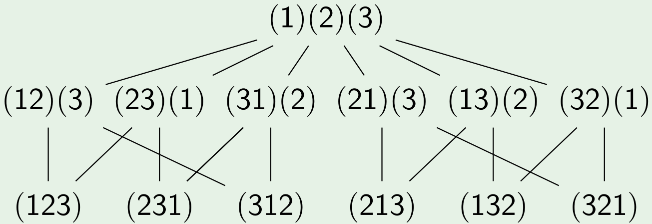
Decorated partition posets [Vallette, 07]

Definition

Let $\mathcal{P}$ be a connected set operad.
A $\mathcal{P}$-decorated partition of a finite set V is an element of $\mathrm{Comm} \circ \mathcal{P}$.
The set of $\mathcal{P}$-decorated partitions of V is endowed with the partial order

$$(\alpha, \eta) \leqslant (\beta, \xi) \Leftrightarrow \alpha \leqslant_{\Pi(V)} \beta, \forall A \in \alpha, \exists \nu_A \in \mathcal{P}(\beta_{|A}) \text{ s.t. } \eta_A = \nu_A \circ (\xi_B)_{B \in \beta_{|A}}.$$

Assoc-decorated partitions of $\{1, 2, 3\}$



Basics

Definition

A set operad $\mathcal{P}$ is

- Left-basic iff $\prod_{T\in\pi}\mathcal{P}(T)\to\mathcal{P}(S)$, $(\xi_T)_{T\in\pi}\mapsto\nu\circ(\xi_T)_{T\in\pi}$ is injective
- Right-basic iff $\mathcal{P}(\pi)\to\mathcal{P}(S)$, $\nu\mapsto\nu\circ(\xi_T)_{T\in\pi}$ is injective

Examples and counter-examples

- Perm is right-basic, but not left-basic.
- The quadratic operad with two generators $\dashv$ and $\vdash$ and the following relations is left-basic but not right-basic.

$$(a\dashv b)\vdash c=(a\dashv b)\dashv c \qquad (a\vdash b)\vdash c=(a\vdash b)\dashv c$$
$$a\vdash(b\dashv c)=a\dashv(b\dashv c) \qquad a\vdash(b\vdash c)=a\dashv(b\vdash c)$$

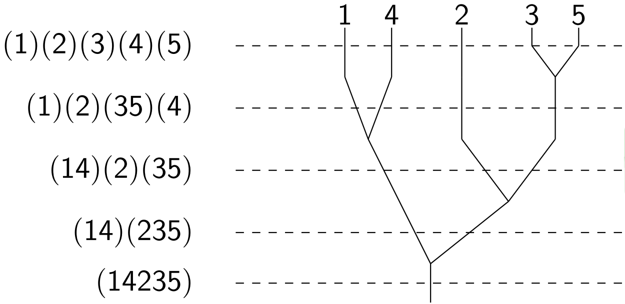
- Assoc and Comm are both left-basic and right-basic.

Decorated partition posets [Vallette, 07]

Theorem (Vallette, 07)

When $\mathcal{P}$ is right-basic, the linear operad $\mathbb{K}\mathcal{P}$ is Koszul iff the associated posets $\Pi^{\mathcal{P}}(V)$ have a unique non trivial cohomology group (Cohen-Macaulay), for any V.
Moreover, in this case, denoting by $(\mathbb{K}\mathcal{P})^!$ its Koszul dual, the unique cohomology group is given by:

$$h^{|V|-1}(\Pi^{\mathcal{P}}(V)) \simeq s^{n-1} (\mathbb{K}\mathcal{P})^! (V) \otimes_{\mathfrak{S}_V} \mathrm{sgn} =: \Lambda^{-1} (\mathbb{K}\mathcal{P})^! (V).$$



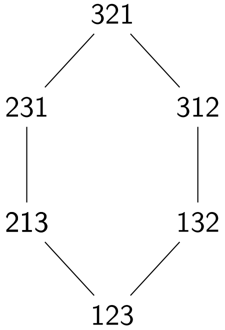
Associative case

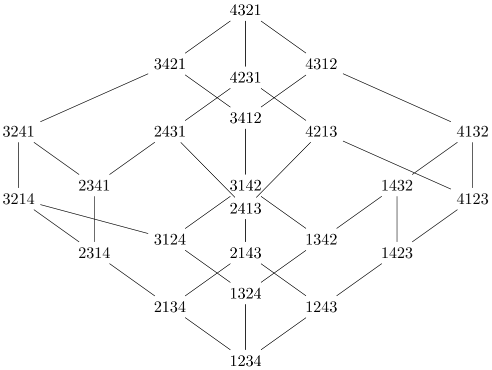
Planar levelled trees.

Permutohedra and braid arrangements

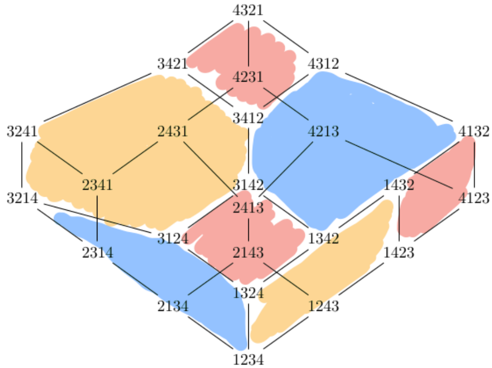
A second example of posets : the weak Bruhat order W_n [Verma 1968]

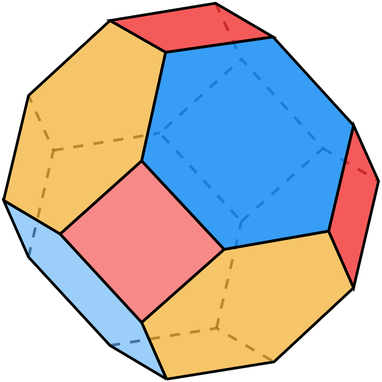
- Covering relations, $\ldots ab \ldots \lhd \ldots ba \ldots$, with $a < b$



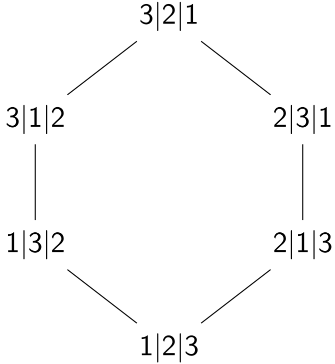


Permutohedron = polytope whose vertices are permutations and whose edges are covering relations in the weak Bruhat order [Schoute 1911]

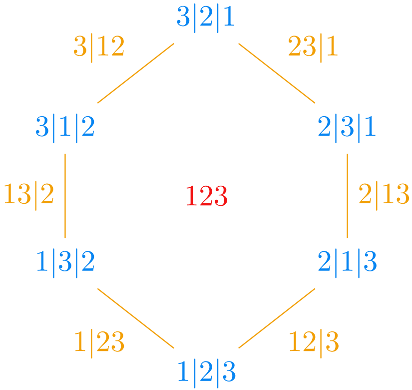




Labelling of faces of the permutohedron



Labelling of faces of the permutohedron



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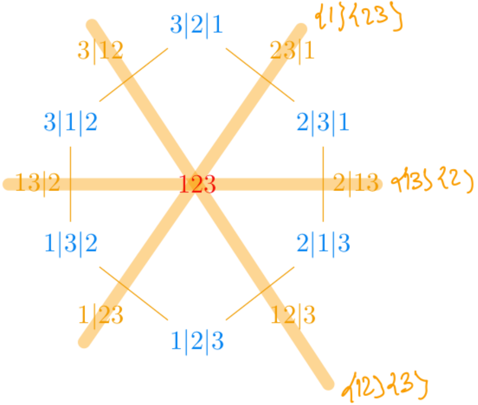
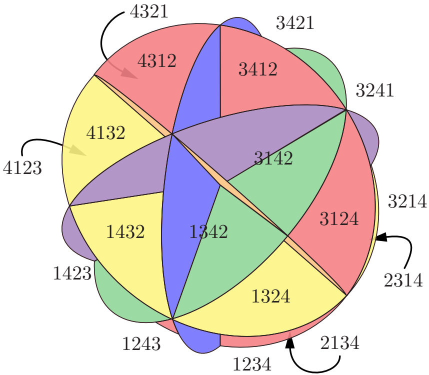
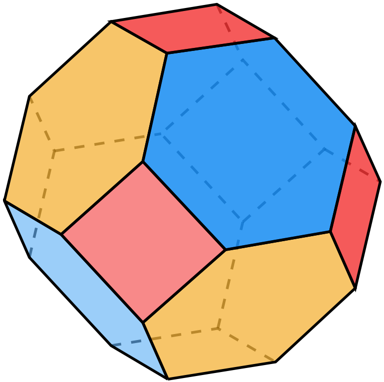


Polytope and hyperplane arrangement





À retenir

Number of faces of dimension k = number of regions of dimension $n - k$

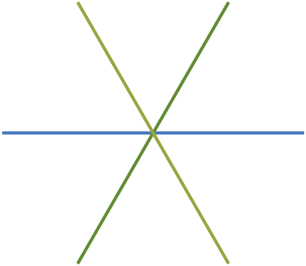
Braid arrangement

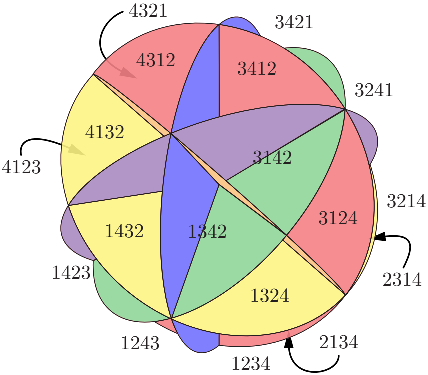
$$H_{i,j}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_i = x_j\}$$

.

$$\mathcal{B}_n = \bigcup_{1 \leqslant i < j \leqslant n} H_{i,j}^n$$

is called the braid arrangement.

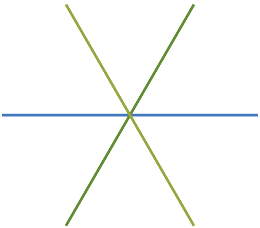




Braid arrangement and set compositions ($^{\mathrm{Assoc}}\Pi$)

Definition

A region of $\mathcal{B}_n$ is a connected component of $\mathbb{R}^n \setminus \bigcup_{1 \leqslant i < j \leqslant n} H^n_{i,j}$. The faces of $\mathcal{B}_n$ are the closures of its regions and all their intersections with one of its hyperplane. Faces are ordered by inclusion.



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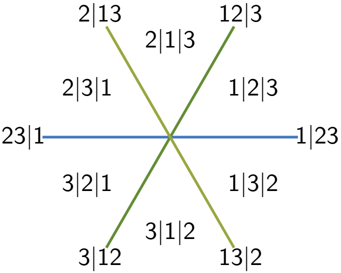

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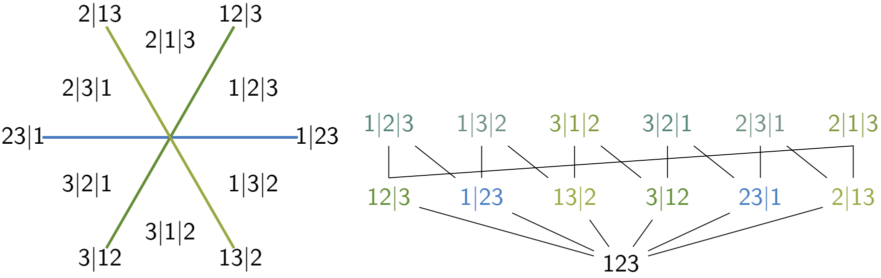
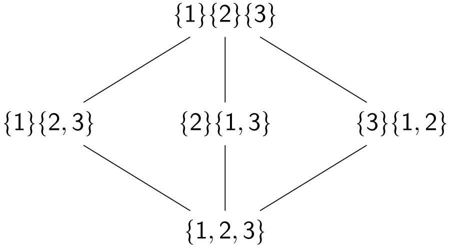
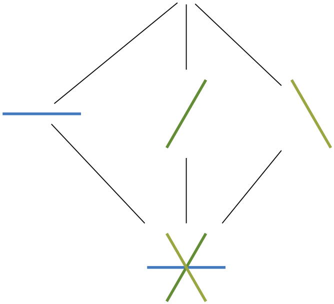


Back to partitions

Definition

An intersection of $\mathcal{B}_n$ is a non-empty affine subspace of $\mathbb{R}^n$ obtained as the intersection of some hyperplane of $\mathcal{B}_n$. Intersections are ordered by containment.





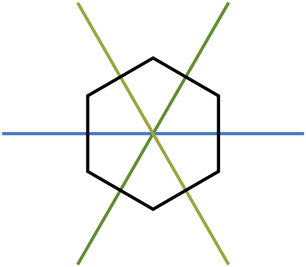
Zaslavsky's theorem

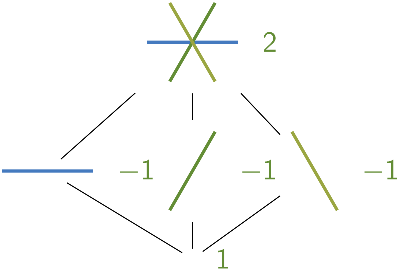
Let $\mathcal{A}$ be a hyperplane arrangement and $\mathcal{I}$ its intersection poset.

Theorem (Zaslavsky, 75)

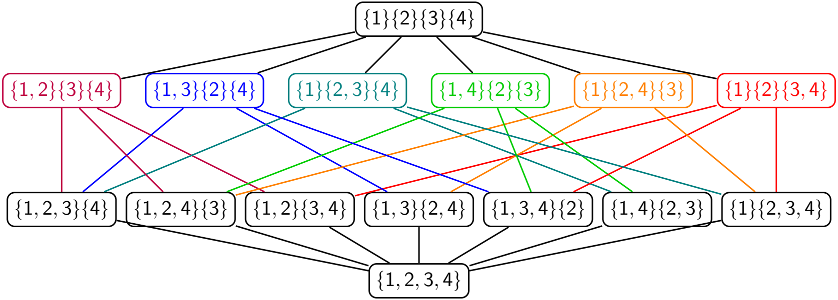
$$\textit{number of } k\textit{-faces} = \sum_{\substack{I \leqslant J \in \mathcal{I} \\ \dim(I) = k}} |\mu(I, J)|,$$

where $\mu(I, J)$ is the Euler characteristic (called Mobius number) of the interval $[I, J]$.





Intervals and Möbius number in partition posets



Lemma

For $\pi = (\pi_1, \ldots, \pi_k) \in \Pi_n$*, we have :*

$$[0_{\Pi_n}, \pi] \simeq \Pi_k \qquad [\pi, 1_{\Pi_n}] \simeq \prod_{i=1}^{k} \Pi_{|\pi_k|} \qquad \mu(0_{\Pi_n}, \pi) = (k-1)!$$

Number of regions of the braid arrangement

Proposition

$$f_k(\mathcal{B}_n) = \sum_{\mathbf{F} \leqslant \mathbf{G}} \prod_{F_i \in \mathbf{F}} (\#\mathbf{G}[F_i] - 1)!$$

where $\mathbf{F} \leqslant \mathbf{G}$ *are two partitions,* $\mathbf{F}$ *with* $k+1$ *parts and* $\mathbf{G}[F_i] = \{G_j \in \mathbf{G} | G_j \subseteq F_i\}$.

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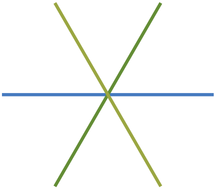
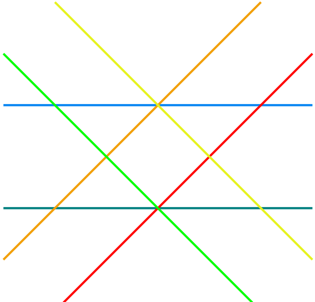
Question

What happens if we consider ℓ copies of the braid arrangement ?



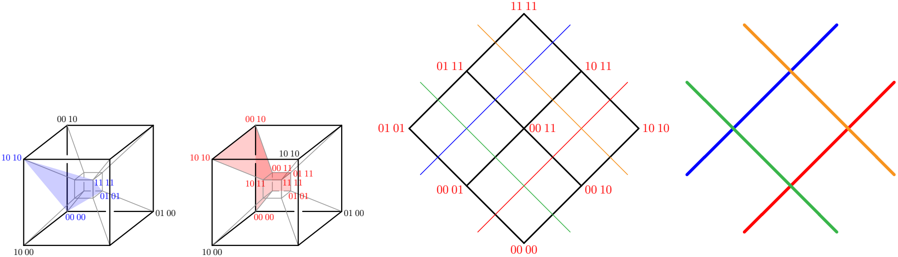


Diagonal of the permutohedron

Motivations

Compute a cellular and coherent version of the thin diagonal $\delta : x \to (x, x)$ of the permutohedron P

More precisely, we define a map $\Delta : P \to P \times P$ whose image is a union of faces of $P \times P$ and homotopic to the thin diagonal.



Applications

Formula for coproducts, tensor product for homotopy operads and description of cup product on Losev-Manin toric varieties.

Number of regions for 2 copies of the braid arrangement

Theorem (BDO, G. Laplante-Anfossi, V. Pilaud, K. Stoeckl)

$$f_{n-k_1-1,n-k_2-1}({\mathcal{B}_n}^2) = \sum_{\mathbf{F} \leqslant \mathbf{G}} \prod_{i \in [2]} \prod_{p \in F_i} (\#G_i[p] - 1)!$$

where **F** *and* **G** *are two forests of* 2*-coloured trees and* $\#F_i = k_i + 1$

$$f_{n-1}({\mathcal{B}_n}^2) = n![x^n]exp\left(\sum_{m \geqslant 1} \frac{x^m}{m(m+1)} \binom{2m}{m}\right) [A213507]$$

$$f_0({\mathcal{B}_n}^2) = 2(n+1)^{n-2}[A007334]$$

$$f_{k,n-k-1}({\mathcal{B}_n}^2) = \frac{1}{k+1}\binom{n}{k}(k+1)^{n-k-1}(n-k)^k$$

Number of regions for ℓ copies of the braid arrangement

Theorem (BDO, G. Laplante-Anfossi, V. Pilaud, K. Stoeckl)

$$f_{n-k_1-1,\dots,n-k_\ell-1}(\mathcal{B}_n{}^\ell) = \sum_{\mathbf{F}\leqslant\mathbf{G}} \prod_{i\in[\ell]} \prod_{p\in F_i} (\#G_i[p]-1)!$$

where **F** *and* **G** *are two forests of ℓ-coloured trees and $\#F_i = k_i + 1$*

$$f_{n-1}(\mathcal{B}_n{}^\ell) = n![x^n]\exp\left(\sum_{m\geqslant 1} \frac{x^m}{m(1+(\ell-1)m)}\binom{\ell m}{m}\right)$$

$$f_0(\mathcal{B}_n{}^\ell) = \ell\,(1+(\ell-1)n)^{n-2}$$

Also

- Combinatorial description of faces of the diagonal
- Only two operadic diagonals on the permutohedron

Operadic poset species

Cohomology of the hypertree poset

Theorem (Conjecture of Chapoton, ; proven in 0.,13)

The augmented hypertree poset $\widehat{HT}(V)$ *is Cohen-Macaulay and*

$$\tilde{H}^{|V|-3}(\widehat{HT}(V) \backslash \{\hat{0}, \hat{1}\}) = \Lambda^{-1}\widehat{\mathsf{PreLie}}(V),$$

for a finite set S *of size* n.

Cohomology of the hypertree poset

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Why do we find an operad here ?

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Question

Why do we find an operad here ?

Answer

Operadic poset species

Properties of the partition posets

Proposition (Folklore)

For every partition $\pi \in \Pi(S)$ *we have isomorphisms of posets*

$$\varphi_\pi : \Pi_{\leqslant \pi}(S) \xrightarrow{\sim} \Pi(\pi) \qquad \text{and} \qquad \psi_\pi : \Pi_{\geqslant \pi}(S) \xrightarrow{\sim} \prod_{T \in \pi} \Pi(T)$$

defined by $\alpha \mapsto \{\pi_{|T}, T \in \alpha\}$ *and* $\beta \mapsto (\beta_{|T})_{T \in \pi}$ *respectively.*

Examples

Let $S = \{a, b, c, d, e, f, g\}$ and $\pi = \{T_1, T_2, T_3\} =: T_1|T_2|T_3$, with $T_1 = \{a, b, c\},\ T_2 = \{d, e\},\ T_3 = \{f, g\}$.

$$\varphi_\pi(x) = \varphi_\pi(abcde|fg) = 12|3 =: x/\pi$$
$$\psi_\pi(a|bc|d|e|fg) = (a|bc, d|e, fg)\,.$$

Composition of cochains

Let S be a finite set and π be a partition of S.
Denoting by Künneth morphisms by κ, we have the following map:

$$c^\bullet(\Pi(\pi)) \otimes \bigotimes_{T \in \pi} c^\bullet(\Pi(T)) \stackrel{id \otimes \kappa}{\to} c^\bullet(\Pi(\pi)) \otimes c^\bullet\left(\prod_{T \in \pi} \Pi(T)\right)$$

$$\stackrel{\varphi_\pi^* \otimes \varphi_\pi^*}{\to} c^\bullet(\Pi_{\leqslant \pi}(S)) \otimes c^\bullet(\Pi_{\geqslant \pi}(S)) \to c^\bullet(\Pi(S)).$$

This does not define a differential graded operad on $c^\bullet$ (associativity and commutativity fail) but it induces a graded operad structure on the cohomology which is exactly $\Lambda^{-1}Lie$.

Operadic poset species

Let P be a poset species, with $a : P \to \Pi$, s.t. for any finite set S, $a(S) : P(S) \to \Pi(S)$ **strictly increasing.**
We consider

$$\varphi_x : P_{\leqslant x}(S) \to P(\pi) \qquad \text{and} \qquad \psi_x : P_{\geqslant x}(S) \to \prod_{T \in \pi} P(T)$$

Definition

The poset species P with a, φ_x and ψ_x is an operadic poset species if

- $\varphi_\pi \circ a = a \circ \varphi_x, \quad \psi_\pi \circ a = a \circ \psi_x$
- φ_x and ψ_x satisfy moreover some equivariance, unitality and associativity axioms.

Theorem (D.O. - Dupont, 24+)

$h^\bullet(P)$ is endowed with a structure of graded operad of $\mathbb{K}$-modules.

Consequences of the construction

Theorem (D.O. - Dupont, 24+)

$h^\bullet(P)$ is endowed with a structure of graded operad of $\mathbb{K}$-modules.

Proof: We construct a morphism
$\rho_\pi : h^\bullet(\Pi(\pi)) \otimes \bigotimes_{T\in\pi} h^\bullet(\Pi(T)) \to h^\bullet(\Pi(S))$ for any $\pi \in \Pi(S)$.

□

Corollary

$h^\bullet(P)$ is equipped with morphism of graded operads $a^* : \Lambda^{-1}Lie \to h^\bullet(P)$.

Counter example

The boolean posets is NOT an operadic poset species.

Other cohomologies

By considering

$$\check{c}^{k}(P) = \mathbb{K}.\{x_0 < \ldots < x_k | x_0 \in \min(P)\}$$

$$\hat{c}^{k}(P) = \mathbb{K}.\{x_0 < \ldots < x_k | x_k \in \max(P)\}$$

we obtain morphisms

$$\check{\rho}_\pi : h^\bullet(P(\pi)) \otimes \bigotimes_{T \in \pi} \check{h}^\bullet(P(T)) \to \check{h}^\bullet(P(S)).$$

$$\hat{\rho}_\pi : \hat{h}^\bullet(P(\pi)) \otimes \bigotimes_{T \in \pi} h^\bullet(P(T)) \to \hat{h}^\bullet(P(S)).$$

Proposition (D.O. - Dupont, 24+)

$\check{h}^\bullet(P)$ is a left operadic module over $h^\bullet(P)$.
$\hat{h}^\bullet(P)$ is a right operadic module over $h^\bullet(P)$.

A bunch of new examples

Operadic poset species P	$h^\bullet(P)$	$\check{h}^\bullet(P)$	$\hat{h}^\bullet(P)$
Π	Lie	-	-
${}^{\mathcal{P}}\Pi$	$(\mathbb{K}\mathcal{P})^!$ (if Koszul)	non-trivial	-
${}^{\mathsf{As}}\Pi$	$(\mathbb{K}\,\mathsf{As})$	$\mathbb{K}\,\mathsf{Com}$	-
$\Pi^{\mathcal{Q}}$	$(\mathbb{K}\mathcal{Q})^!$ (if Koszul)	-	non-trivial
Π^{As}	$(\mathbb{K}\,\mathsf{As})^!$	-	not Cohen-Macaulay
Π^{Perm}	$(\mathbb{K}\,\text{pre-Lie})^!$	-	? [**?**, A000312]
2-NCP	? [**?**, A000312]	?	-
NS	MetabLie	-	-
MLCT	Graph complex	?	-
HT	post-Lie	pre-Lie	-

Wishlist

- Study the cyclic operad structure on the cohomology.
- Define directly the operadic poset structure in terms of nested sets associated with the minimal building set [cf. work of B. Coron]
- Other examples ? (for instance bidecorated partitions and bidecorated hypertrees)

Wishlist

- Study the cyclic operad structure on the cohomology.
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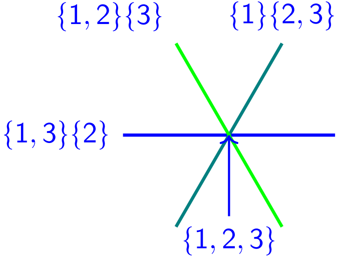
- Other examples ? (for instance bidecorated partitions and bidecorated hypertrees)

Thank you for your attention !

Description of intersections in terms of trees



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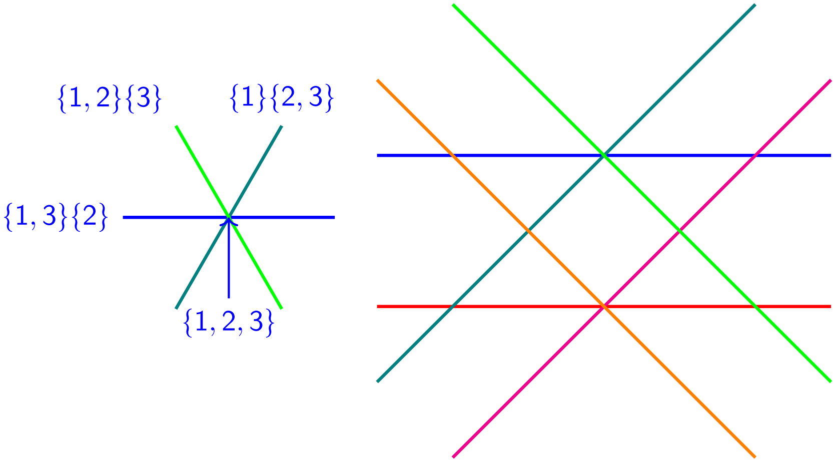

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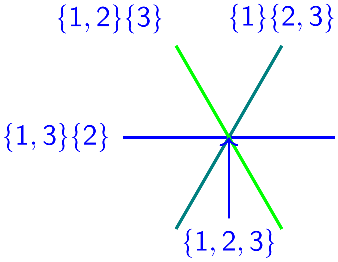

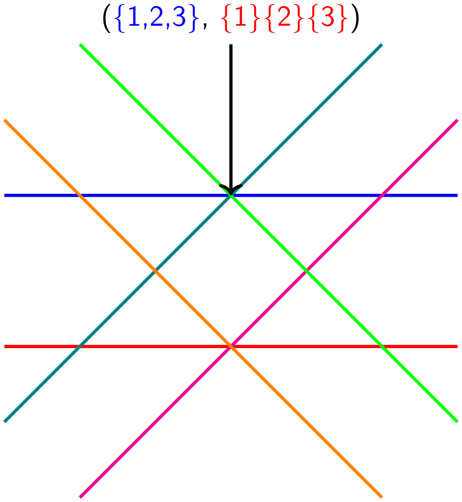

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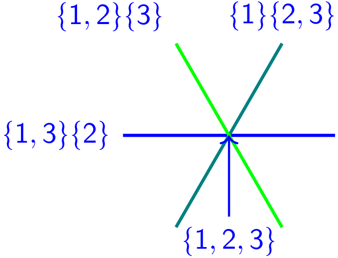

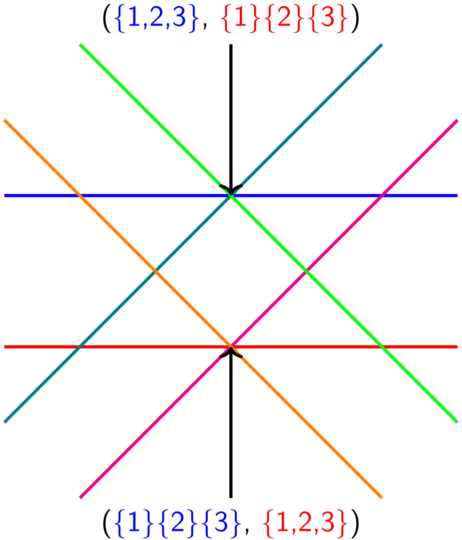

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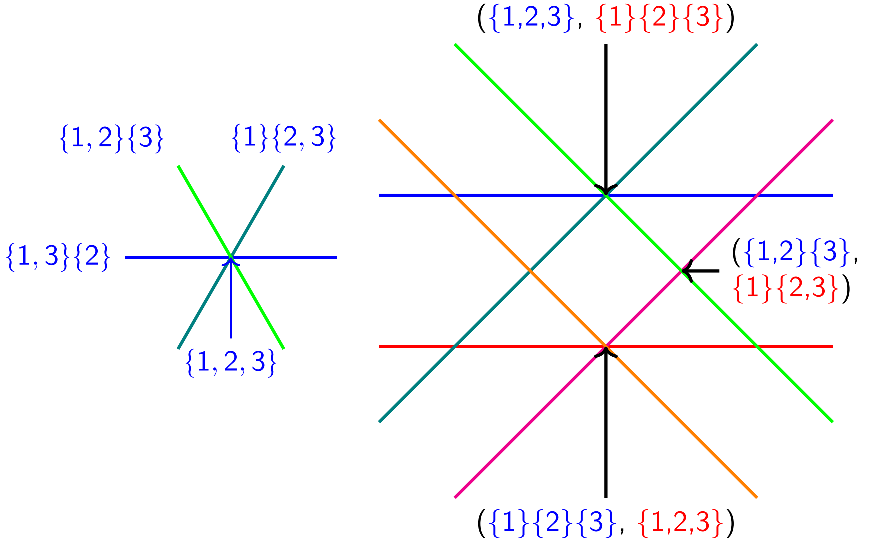

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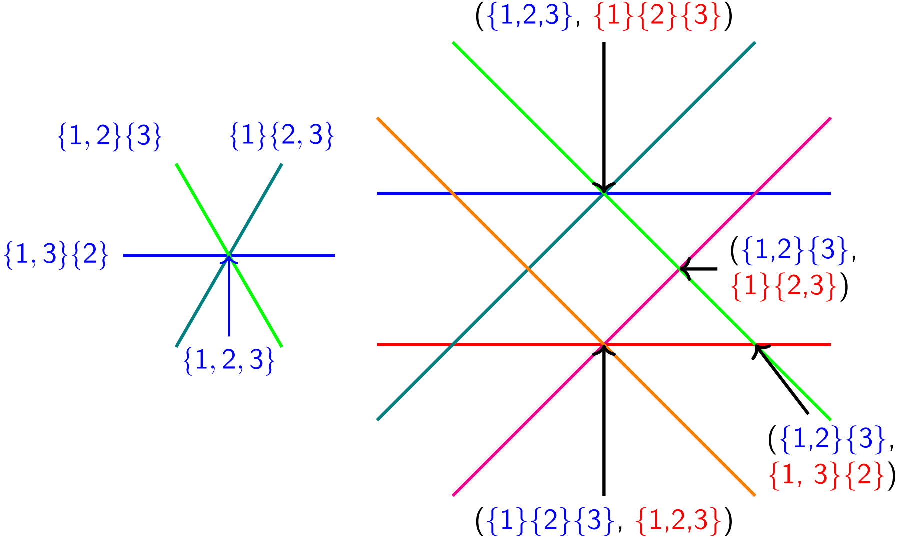

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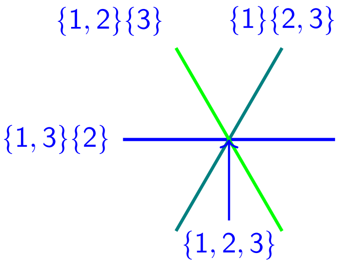

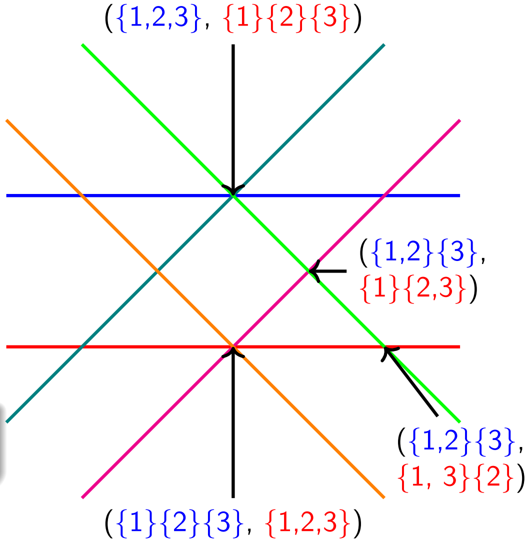

Some pairs don't appear

~~({1, 2}{3}, {1, 2}{3})~~

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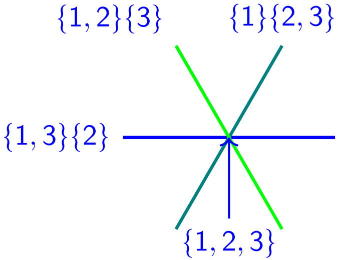

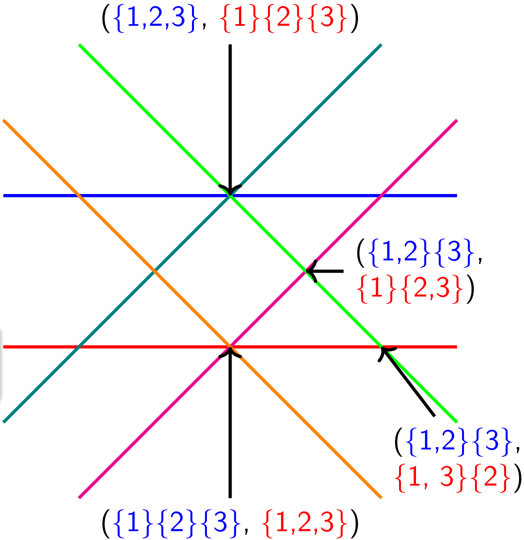

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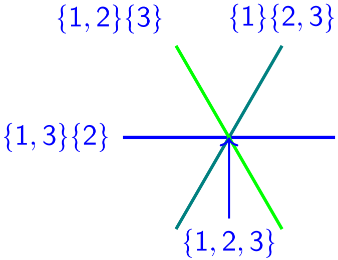
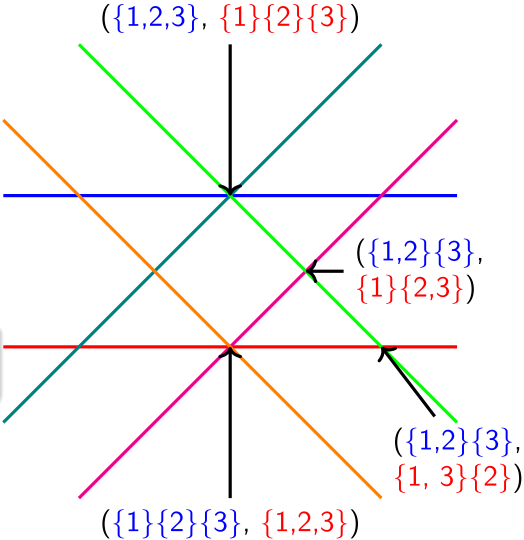
1 2 1

3 23

Description of intersections in terms of trees





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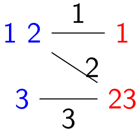

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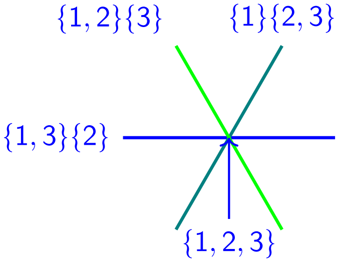

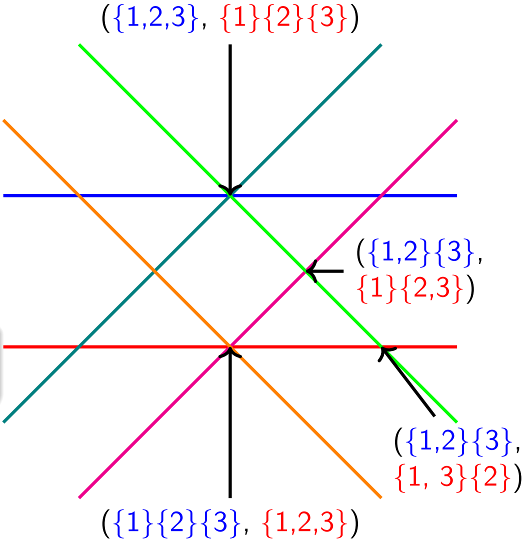

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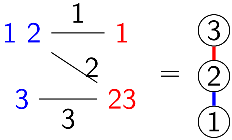
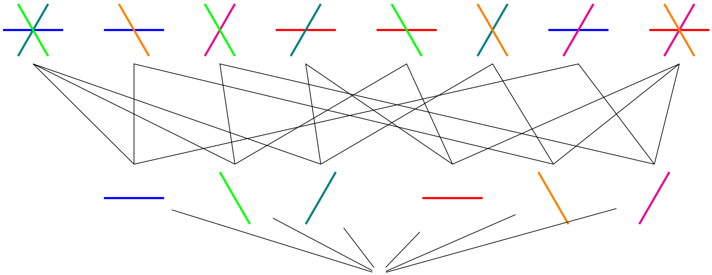


De l'intersection d'hyperplans aux forêts colorées

Intersection d'hyperplans

Chaque intersection est une forêts d'arbres enracinés aux arêtes colorées telles que :

- il y a ℓ couleurs d'arêtes différentes et 1 est une racine,
- L'arête partant d'un enfant n'a pas la même couleur que l'arête le reliant à son parent.



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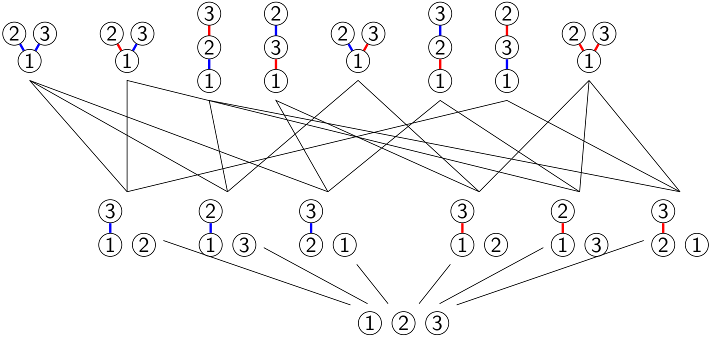


First example : Right-decorated partitions posets $\Pi^{\mathcal{P}}$ aka Vallette's generalised partition posets

- $a(\pi, \xi) = \pi$
- $\varphi_{(\pi,\xi)}((\alpha, \eta)) = (\alpha/\pi, \nu)$ (recalling $\eta_A = \nu_A \circ (\xi_P)_{P \in \pi_{|A}}$ for any part A of α): it is NOT an isomorphism.
- $\psi_{(\pi,\xi)}((\beta, \nu)) = \prod_{T \in \pi} \beta_{|T}$: it is an isomorphism of posets.

Proposition (D.O. - Dupont, 24+)

$\Pi^{\mathcal{P}}$ *is an operadic poset species.*

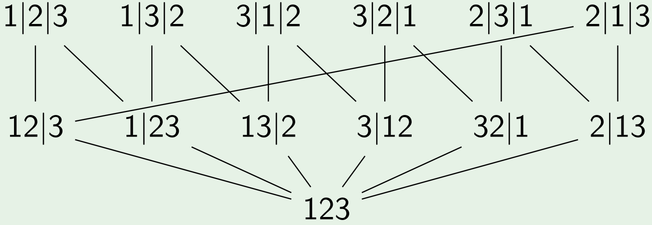
Second example : Left-decorated partitions posets ${}^{\mathcal{P}}\Pi$

Definition

Let $\mathcal{P}$ be a set operad satisfying $\mathcal{P}(\emptyset) = \emptyset$ and $\mathcal{P}(\{*\}) = \{*\}$.
A Left-$\mathcal{P}$-decorated partition of a finite set V is a pair (π, ξ), where π is a partition of V and $\xi \in \mathcal{P}(\pi)$.
The set of Left-$\mathcal{P}$-decorated partitions of V is endowed with the partial order

$$(\alpha, \nu) \leqslant (\beta, \eta) \Leftrightarrow \alpha \leqslant_{\Pi(V)} \beta, \eta = \nu \circ (\xi_A)_{A \in \alpha}.$$

${}^{\mathrm{Assoc}}\Pi(\{1,2,3\})$, aka Face poset of the permutohedron



Second example : Left-decorated partitions posets ${}^{\mathcal{P}}\Pi$

- $a(\pi, \xi) = \pi$
- $\varphi_{(\pi,\xi)}((\alpha, \eta)) = (\alpha/\pi, \tilde{\eta})$, where $\tilde{\eta}$ is de decoration of $\mathcal{P}(\alpha/\pi)$ induced by η: it is an isomorphism.
- $\psi_{(\pi,\xi)}((\beta, \eta)) = \prod_{T \in \pi}(\beta_{|T}, \mu_T)$, where $\eta = \xi \circ (\mu_T)_{T \in \pi}$: it is NOT an isomorphism of posets.

Proposition (D.O. - Dupont, 24+)

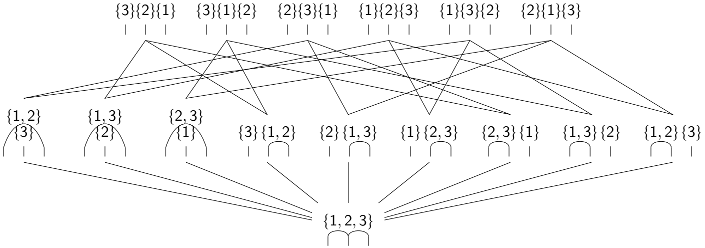
When $\mathcal{P}$ is left-basic, ${}^{\mathcal{P}}\Pi$ is an operadic poset species.

First example : parking function

Definition

Given a finite set S, a S-parking function is

- a non-crossing partition $\pi = (\pi_1, \ldots, \pi_k)$ (where we order the parts according to their minimal elements) of $\{1, \ldots, |S|\}$,
- whose parts are labeled by a subset of S of same size,
- so that the labels form a partition of S,



Proposition (DO–Josuat-Vergès–Randazzo, 22; Kreweras, 72)

For any finite set S, the poset $\Pi_2(S) \cup \hat{1}$ with an added maximum and the maximal intervals of $\Pi_2(S)$ are shellable, hence Cohen–Macaulay.

$$\dim h^{n-1}(\Pi_2(\{1, \ldots, n\})) = n! C_n = (2n-2)(2n-1)\ldots n,$$

where C_n is the nth Catalan number. As an $\mathfrak{S}_n$-module, it is made of C_n copies of the regular representation.

Proposition

The poset species Π_2 is an operadic poset species.

Proposition

We have the equality in $h^2(\Pi_2(3))$:

$$(1 \prec 2) \prec 3 + 1 \prec (2 \prec 3) + (1 \prec 3) \prec 2 + 1 \prec (3 \prec 2) = 0.$$

In particular, the map $a^* : \Lambda^{-1}\mathrm{Lie} \to h^\bullet(\Pi_2)$ factors through $\Lambda^{-1}\mathrm{PreLie}$.

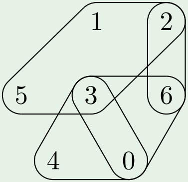
Hypergraphs

Definition (Berge)

A hypergraph (on a set V) is an ordered pair (V, E) where:

- V is a finite set (vertices)
- E is a collection of subsets of cardinality at least two of elements of V (edges).

Example of a hypergraph on $[1; 7]$



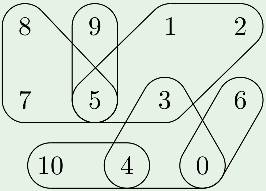
Hypertrees

Definition

A hypertree is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

- there exists a walk from v to w in H with distinct edges e_i, (H is connected),
- and this walk is unique, (H has no cycles).

Example of a hypertree



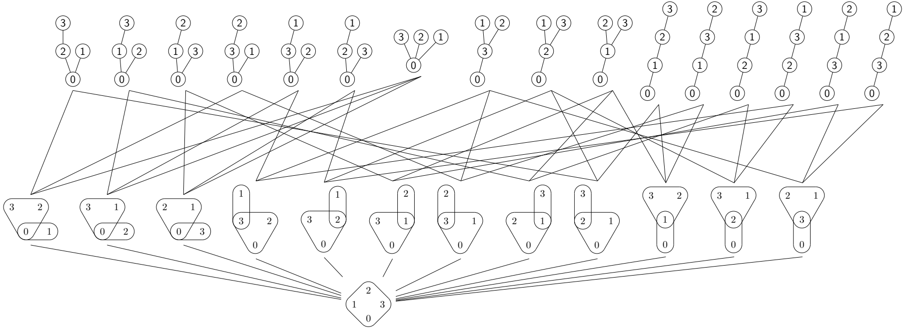
The hypertree poset

Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.



Euler characteristic of the hypertree posets

Proposition (McCammond-Meier, 2004)

The dimension of the top cohomology group of $\widehat{\mathrm{HT}}_n$ *is given by:*

$$\dim\left(H^{n-2}(\widehat{\mathrm{HT}}_n)\right) = (-1)^{n-1}(n-1)^{n-2}$$

Proposition

The dimension of the top cohomology group of HT_n *is given by:*

$$\dim\left(H^{n-2}(\mathrm{HT}_n)\right) = (-1)^n \frac{(2n-3)!}{(n-1)!}$$

$$\frac{(2n-3)!}{(n-1)!} \; ?$$

A006963	Number of planar embedded labeled trees with n nodes: (2n-3)!/(n-1)! for n >= 2, a(1) = 1. (Formerly M3076)	28

1, 1, 3, 20, 210, 3024, 55440, 1235520, 32432400, 980179200, 33522128640, 1279935820800, 53970627110400, 2490952020480000, 124903451312640000, 6761440164390912000, 393008709555221760000, 24412776311194951680000, 1613955767240110694400000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS
For n>1: central terms of the triangle in A173333; cf. A001761, A001813. - Reinhard Zumkeller, Feb 19 2010
Can be obtained from the Vandermonde permanent of the first n positive integers; see A093883. - Clark Kimberling, Jan 02 2012
All trees can be embedded in the plane, but "planar embedded" means that orientation matters but rotation doesn't. For example, the n-star with n-1 edges has n! ways to label it, but rotation removes a factor of n-1. Another example, the n-path has n! ways to label it, but rotation removes a factor of 2. - Michael Somos, Aug 19 2014

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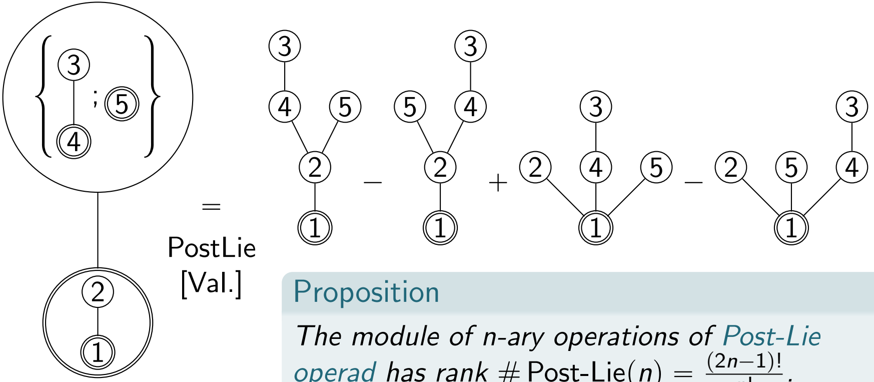
Pierre Leroux and Brahim Miloudi, Généralisations de la formule d'Otter, Ann. Sci.

Post-Lie operad [Vallette, 07 ; Munthe-Kaas–Wright, 08]

The underlying module PostLie(V) of post-Lie operad is spanned by Lie brackets of planar trees with nodes labeled by V. The substitution of a tree t inside a node v is given by the sum over all the way to graft each child of v to the right of a node of t (planar pre-Lie product).



Proposition

The module of n-ary operations of Post-Lie operad has rank $\#\,\text{Post-Lie}(n) = \frac{(2n-1)!}{n!}$.

The hypertree poset species is an operadic poset species

Let H be a hypertree on S and E' be the set of edges of H without their closest vertex to 0.

- $a(H) = E'$
- $\varphi_H(G) =$hypertree induced by G on $S/V(H)$
- $\psi_H(J) = \prod_{e \in E'} J_{|e}$

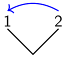
Proposition (D.O. - Dupont, 24+)

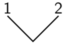
HT is an operadic poset species.

Operadic structure on the cohomology of the nested set complex (aka. post-Lie !)

Let us consider the map

$$\text{Post-Lie} \xrightarrow{\phi} h^\bullet(HT_\bullet)$$

$$1 \lhd 2 \mapsto$$ 

$$\{1;2\} \mapsto$$ 

Theorem (DO–Dupont, 22+)

The map ϕ is an operad morphism. The operadic structure on the cohomology of the hypertree posets is then the desuspension of post-Lie operad.

Left operadic module structure

By considering chains from the minimal element to anywhere, we prove that preLie operad as a left post-lie module structure.

$$
\begin{aligned}
1 \triangleleft T &= 1 \leftharpoondown T, \\
(G \leftharpoondown D) \triangleleft T &= (G \triangleleft T) \leftharpoondown D + G \leftharpoondown (D \triangleleft T) \\
\{S, T\} &= T \leftharpoondown S - S \leftharpoondown T,
\end{aligned}
$$

where $\leftharpoondown$ is the usual pre-Lie product.

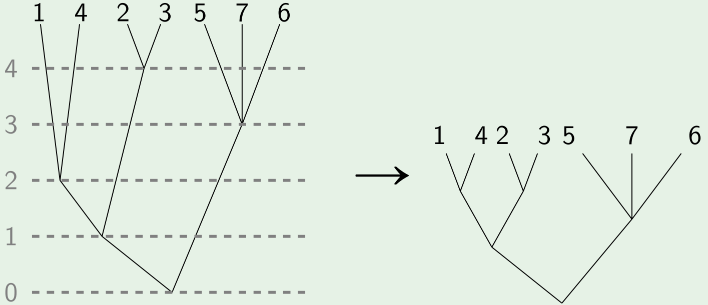
Nested sets

Problem

There are no operadic structure on the leveled cobar construction, but there is one on the cobar construction !

Solution :

Forget about the levels !



This is what we obtain when we consider nested sets instead of chains !

Building sets and nested sets [De Concini–Procesi, 95 ; Feichtner–Müller, 05]

Consider $\mathcal{L}$ a finite join-semilattice (any nonempty subset has a least upper bound). For any $S \subseteq \mathcal{L}$ and $x \in \mathcal{L}$, we write

$$S_{\geqslant X} = \{y \in S | y \geqslant x\}.$$

Definition

A building set is a subset $\mathcal{G}$ in $\mathcal{L}_{<\hat{1}}$ such that for any $x \in \mathcal{L}_{<\hat{1}}$ and $\max \mathcal{G}_{\geqslant x} = \{g_1, \ldots, g_k\}$,there is an isomorphism of posets

$$[x, \hat{1}] \simeq \prod_{i=1}^{k} [g_i, \hat{1}].$$

A nested set is a subset S of $\mathcal{G}$ such that for any set of incomparable elements $x_1, \ldots, x_t$ in S ($t \geqslant 2$), the set $\{x_1, \ldots, x_t\}$ has a greatest lower bound (meet) which does not belong to $\mathcal{G}$.

Topological result

The $\mathcal{G}$-nested sets form an abstract simplicial complex, called the nested set complex.

Proposition (Feichtner–Müller, 05)

Consider a join-semilattice $\mathcal{L}$ and an associated building set $\mathcal{G}$. The associated nested set complex is homotopy equivalent to the order complex of the poset.

For partition posets

The cobar resolution (for the Commutative operad) corresponds to the cochain complex of the nested set complex associated with the minimal building set.

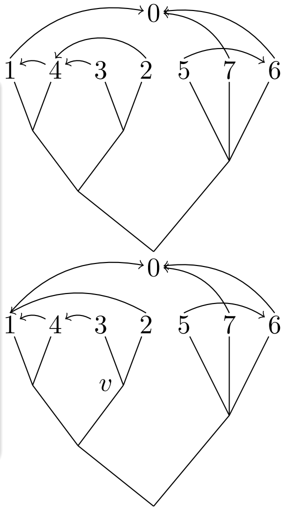
The nested set complex of hypertrees

- Maximal intervals in the hypertree posets are join-semilattices
- The nested sets of hypertrees are the following combinatorial objects:

Definition

A merge tree is a pair (T, τ) of trees such that

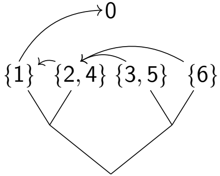
- T is a (non planar) rooted reduced (no vertex of valency 2) tree with leaves labeled by $\{1, \ldots, n\}$
- τ is a (non planar oriented) tree whose vertices are labeled by $\{0, \ldots, n\}$ and whose root is 0
- for any internal vertex s in T, the restriction of τ to edges leaving the leaves above s is connected

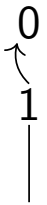


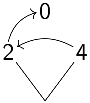
Operadic composition

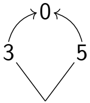
The operadic composition of a bitree b in a node v is as follows:

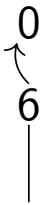
- the blue children of v are grafted to some nodes in b (pre-Lie composition)
- the bottom tree of b is grafted at the place of the leaf v (usual magmatic composition)

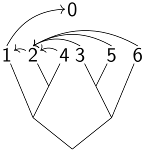
$m_\pi =$ 

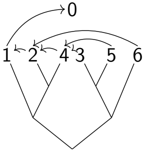
, $m_{\{1\}} =$ 

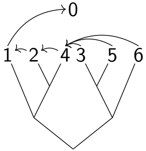
, $m_{\{2,4\}} =$ 

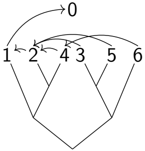
, $m_{\{3,5\}} =$ 

, $m_{\{6\}} =$ 

$\gamma(t(m_\pi), (t(m_p))_{p\in\pi}) =$ 

$+$ 

$+$ 

$+$ 

Operadic structure on the cohomology of the nested set complex (aka. post-Lie !)

Let us consider the map

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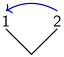
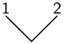
$1 \lhd 2 \mapsto$ 

$\{1;2\} \mapsto$ 

Theorem (DO–Dupont, 22+)

The map ϕ is an operad morphism. The cohomology of the hypertree poset can be endowed with an operadic structure. It is then isomorphic to the suspension of post-Lie operad.