De la diagonale du permutoèdre aux arbres k-colorés :

une histoire de partitions et d'arbres

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joint work with
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Kurt Stoeckl (Univ. Melbourne)

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https://oger.perso.math.cnrs.fr/expose/GDRIM_Oger.pdf

Motivation

algebraic problem : study the diagonal of the permutohedron

↓

geometric problem : counting regions in an hyperplane arrangement

↓

combinatorics problem : counting "good" tuples of partitions

↓

graph problem : counting trees with colored edges

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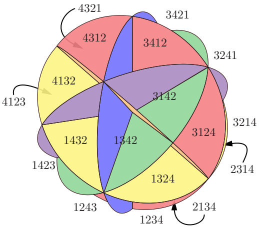
graph problem : counting trees with colored edges

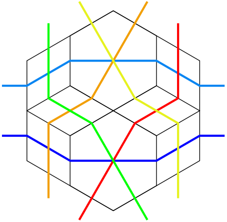
(Yes, combinatorics is mainly counting)

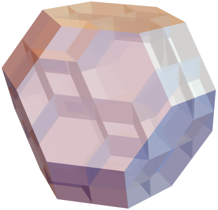
Outline

Trailer

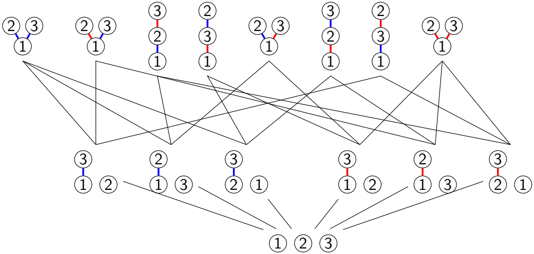


ⒸV. Pilaud





ⒸG. Laplante-Anfossi



The weak order and the permutohedron

Poset=partially ordered set

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First example of poset

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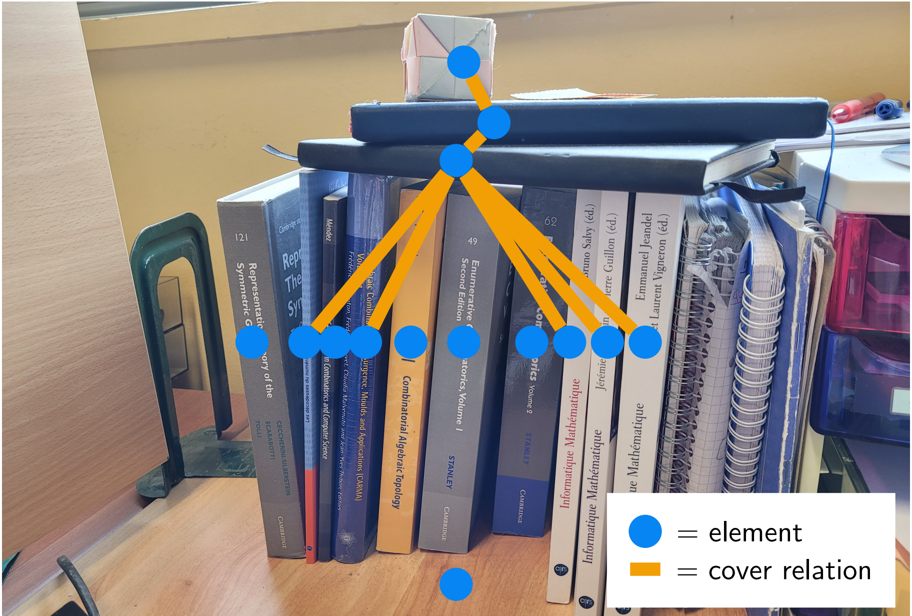

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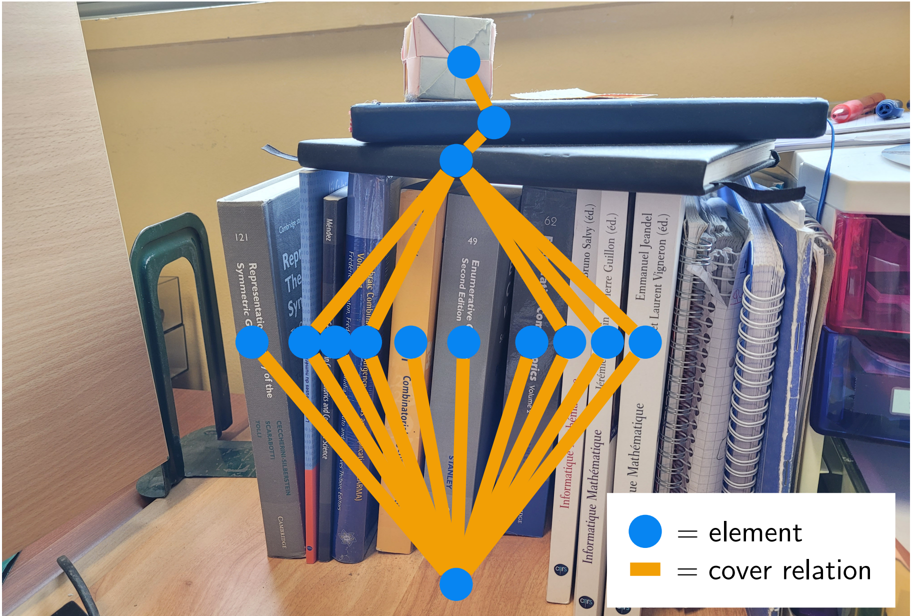

First main example : Weak order W_n

- To raise in the order, $\dots ab \dots \to \dots ba \dots$, with $a < b$

$$123$$

First main example : Weak order W_n

- To raise in the order, $\ldots ab \ldots \rightarrow \ldots ba \ldots$, with $a < b$

123

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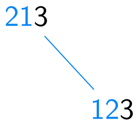

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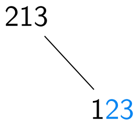

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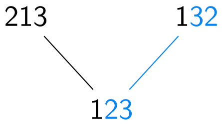

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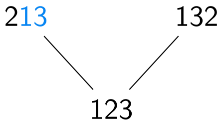

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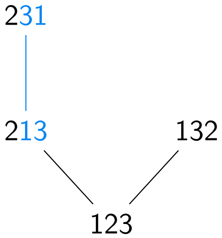

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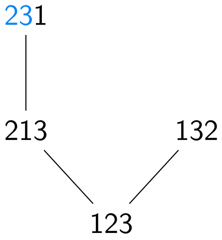

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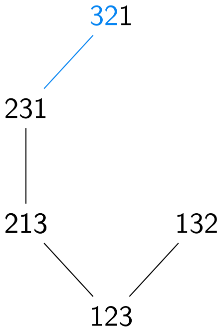

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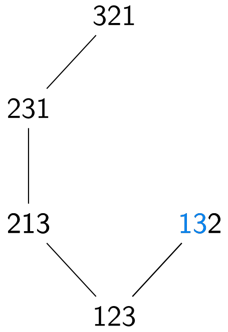

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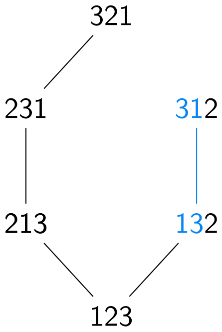

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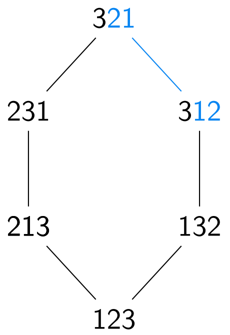

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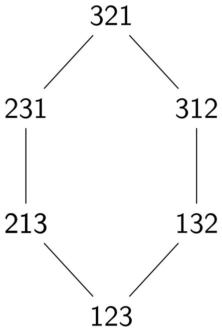

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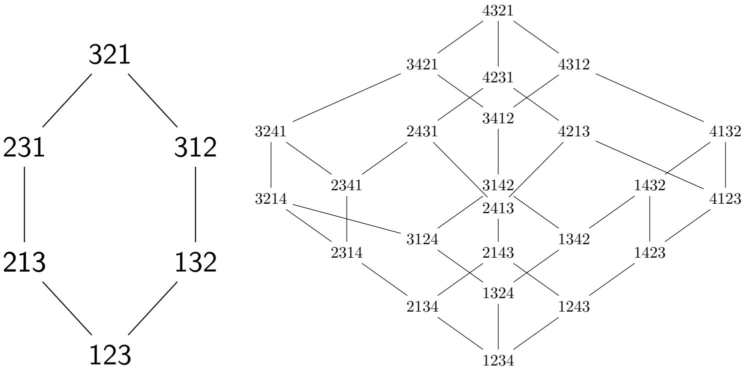
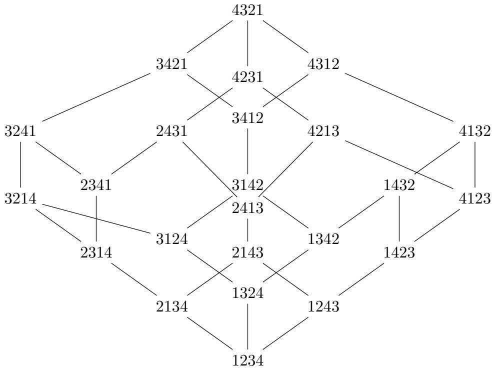
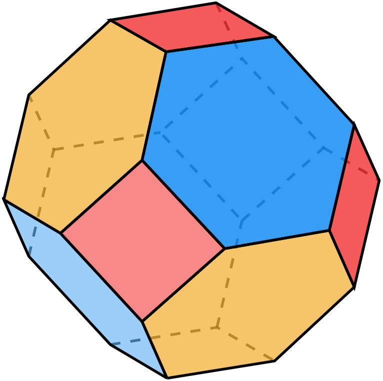


The permutohedron = polytope with vertices labelled by permutation and edges given by the weak order





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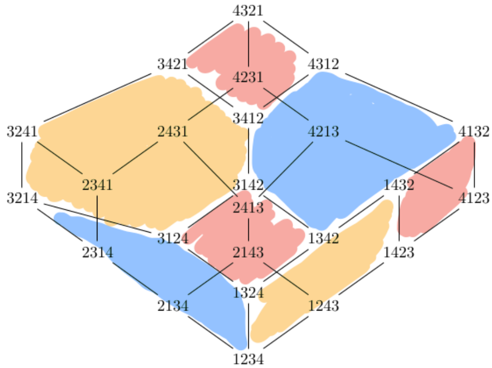

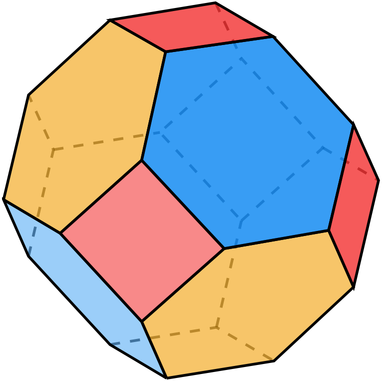

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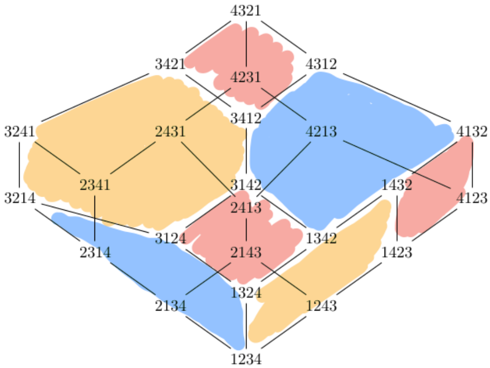

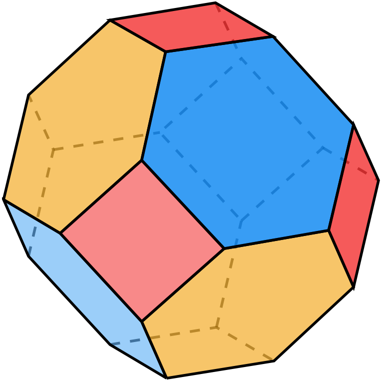

Short quizz :

How many vertices does the permutohedron have ? $n!$! ← Exclamation point

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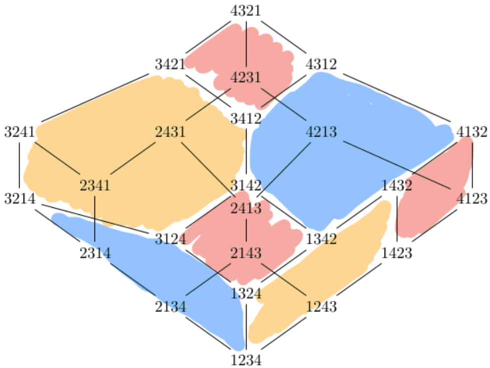
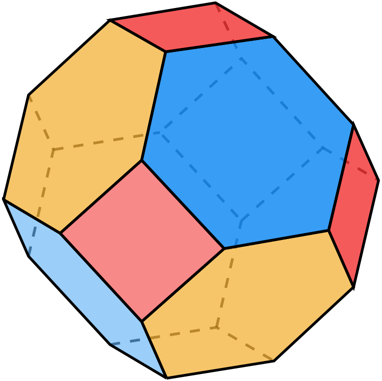




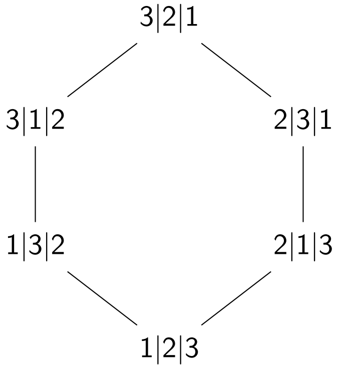
Short quizz :

How many vertices does the permutohedron have ? $n!$!← Exclamation point

How many faces of dimension $n - k$ does the permutohedron have ?

$k!S_2(n, k)$ = nb of ordered partitions in k parts of $\{1, \ldots, n\}$

Labelling of the faces of the permutohedron



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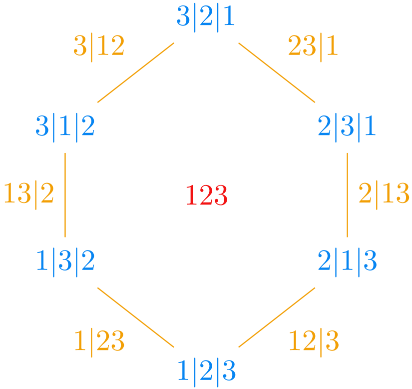

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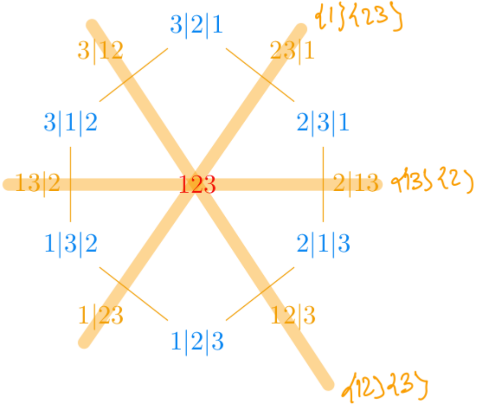
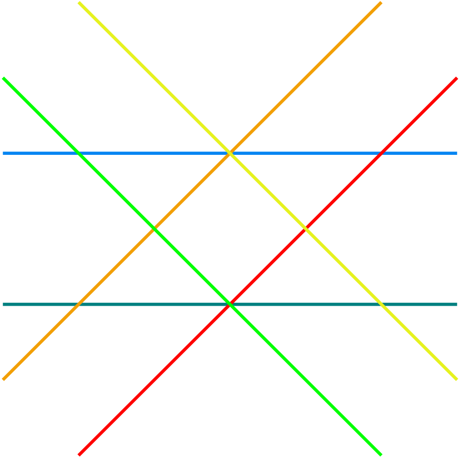
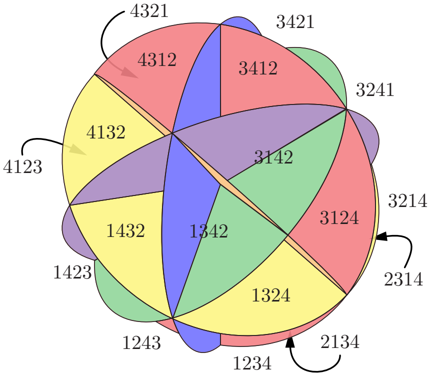


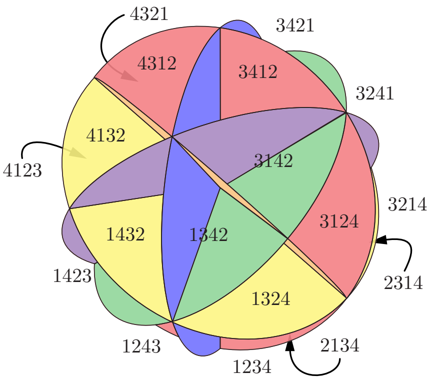
Hyperplane arrangement (Thank you Sylvie !)

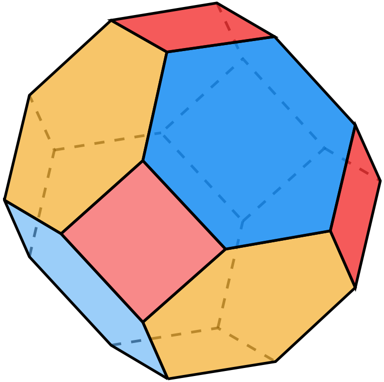
Hyperplane arrangement = set of intersecting affine subspaces of codimension 1





Polytope and hyperplane arrangement





WYMR

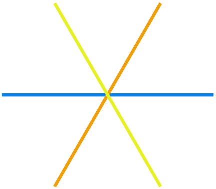
Number of faces of dimension k = number of regions of dimension $n - k$
(linked with Möbius numbers of the intersection poset)

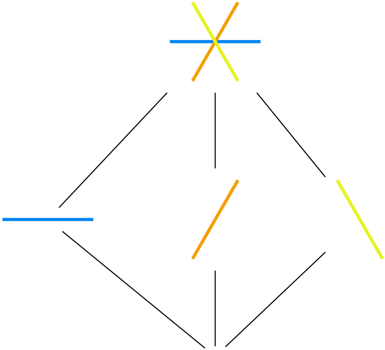
How can we count regions of an hyperplane arrangement ?

Intersection poset

Definition

Intersection poset = Poset of intersections of hyperplanes ordered by (reverse) inclusion





Intersection poset : Another more complicated example

Definition

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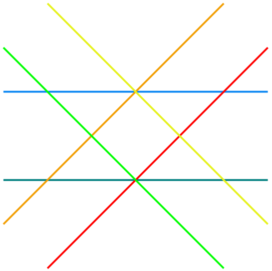

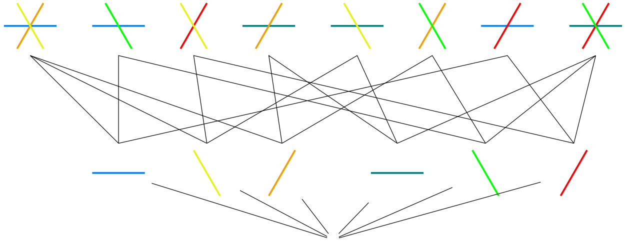

Möbius numbers

Definition

Möbius function : $\mu(x,x) = 1$ and $\mu(x,y) = -\sum_{x \leqslant z < y} \mu(x,z)$

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Just like a game on an oriented graph !

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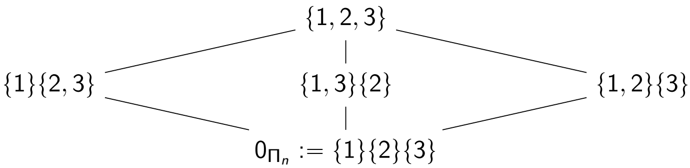

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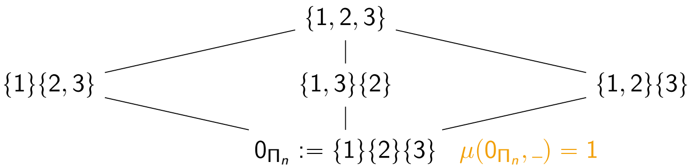

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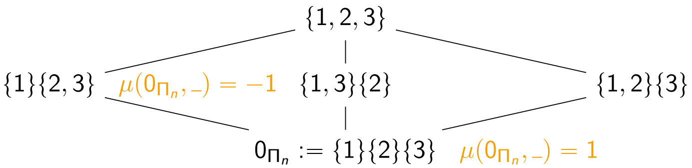

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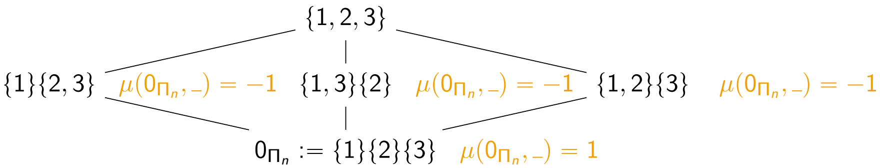

Möbius numbers

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$\{1,2,3\}$ $\mu(0_{\Pi_n}, \_) = 2 \leftarrow$ Möbius number

$\{1\}\{2,3\}$ $\mu(0_{\Pi_n}, \_) = -1$ $\{1,3\}\{2\}$ $\mu(0_{\Pi_n}, \_) = -1$ $\{1,2\}\{3\}$ $\mu(0_{\Pi_n}, \_) = -1$

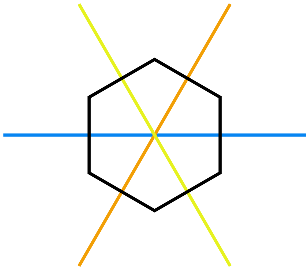
$0_{\Pi_n} := \{1\}\{2\}\{3\}$ $\mu(0_{\Pi_n}, \_) = 1$

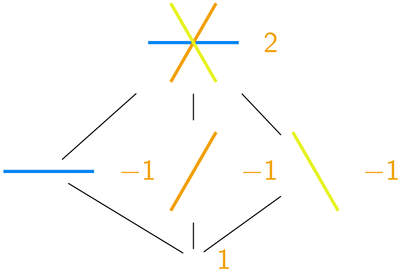
Zaslavsky's theorem

Let $\mathcal{A}$ be an hyperplane arrangement and $\mathcal{I}$ be its intersection poset.

Theorem (Zaslavsky, 75)

$$\textit{number of } k\textit{-faces} \;=\; \sum_{\substack{I \leqslant J \in \mathcal{I} \\ \dim(I) = k}} |\mu(I, J)|$$



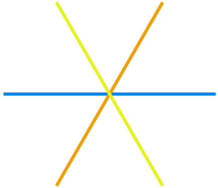


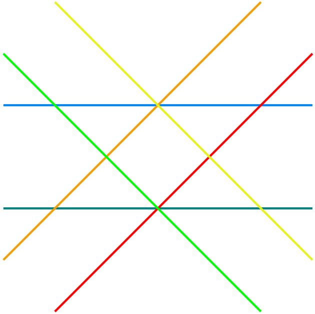
In this talk : ℓ copies of the braid arrangement

Definition

The braid arrangement is the hyperplane arrangement whose hyperplane satisfy equations

$$H_{i,j} = \{x \in \mathbb{R}^n | x_i = x_j\}$$





Intersection poset of the braid arrangement : the partition poset Π_n

Partitions of a set V :

$$\{V_1, \dots, V_k\} \models V \Leftrightarrow V = \bigsqcup_{i=1}^{k} V_i, V_i \cap V_j = \varnothing \text{ for } i \neq j$$

Partial order on set partitions of a set V :

$$\{V'_1, \dots, V'_p\} \leqslant \{V_1, \dots, V_k\} \Leftrightarrow \forall i \in \{1, p\}, \exists j \in \{1, k\} \text{ s.t. } V'_i \subseteq V_j$$

Intersection poset of the braid arrangement : the partition poset Π_n

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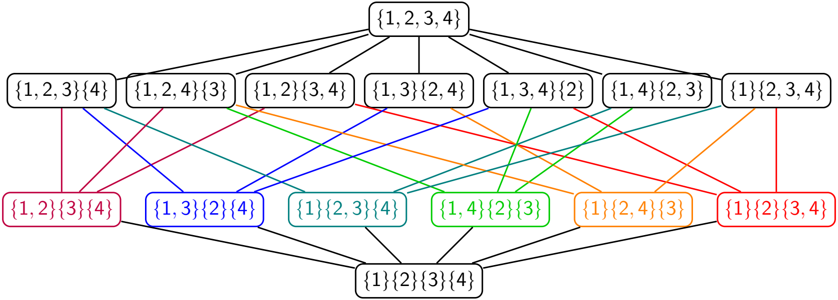
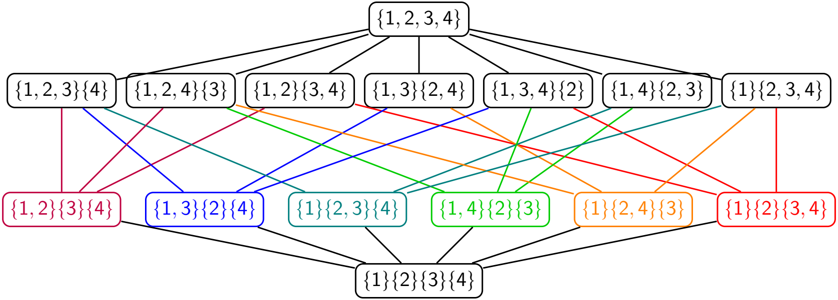


Intervals and möbius numbers of the partition posets



Lemma

For $\pi = (\pi_1, \dots, \pi_k) \in \Pi_n$*, we have :*

$$[0_{\Pi_n}, \pi] \simeq \prod_{i=1}^{k} \Pi_{|\pi_k|} \qquad [\pi, 1_{\Pi_n}] \simeq \Pi_k \qquad \mu(\pi, 1_{\Pi_n}) = (k-1)!$$

Formula for the number of regions of the braid arrangement

Proposition

$$f_k({\mathcal{B}_n}^{\ell}) = \sum_{\mathbf{F} \leqslant \mathbf{G}} \prod_{G_i \in \mathbf{G}} (\#\mathbf{F}[G_i] - 1)!$$

where $\mathbf{F} \leqslant \mathbf{G}$ *are two partitions,* $\mathbf{F}$ *has* $k+1$ *parts and* $\mathbf{F}[G_i] = \{F_j \in \mathbf{F} | F_j \subseteq G_i\}$

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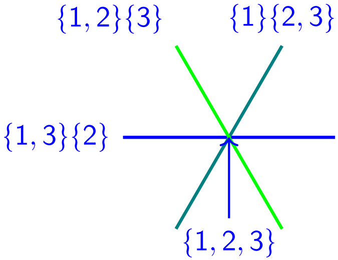
Focus of the next section

What are the underlying combinatorial object when $\ell \geqslant 2$?

The section for which you can wake up if you love graphs but hate algebra

Description of faces in terms of trees



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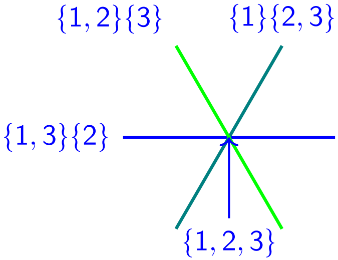
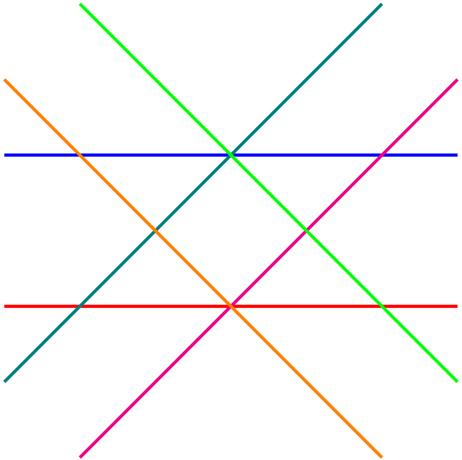
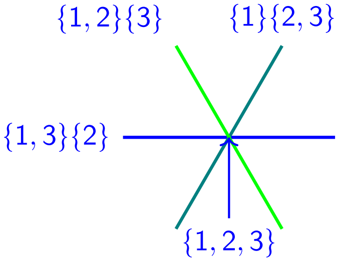
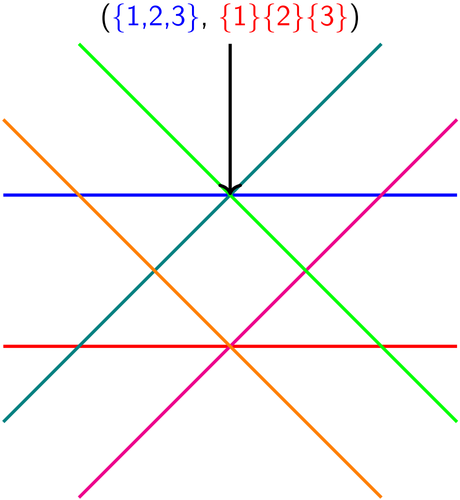




Description of faces in terms of trees





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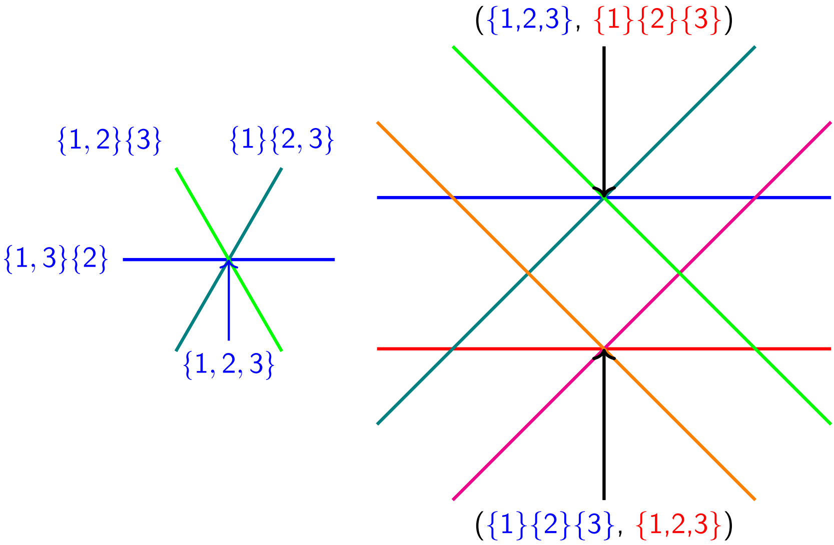

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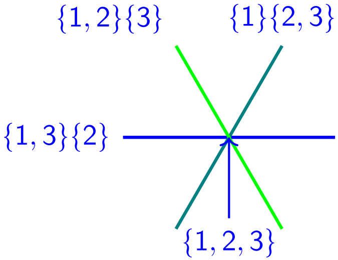

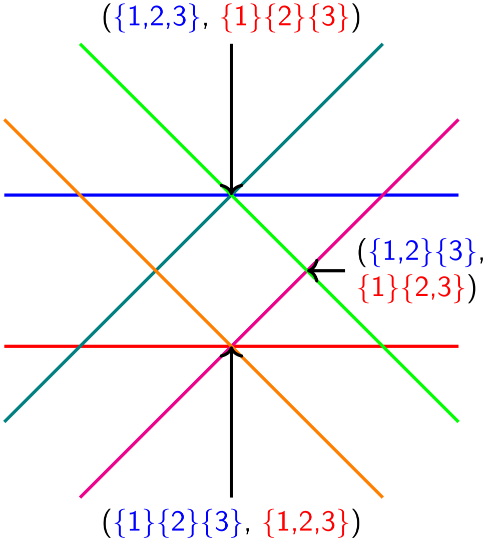

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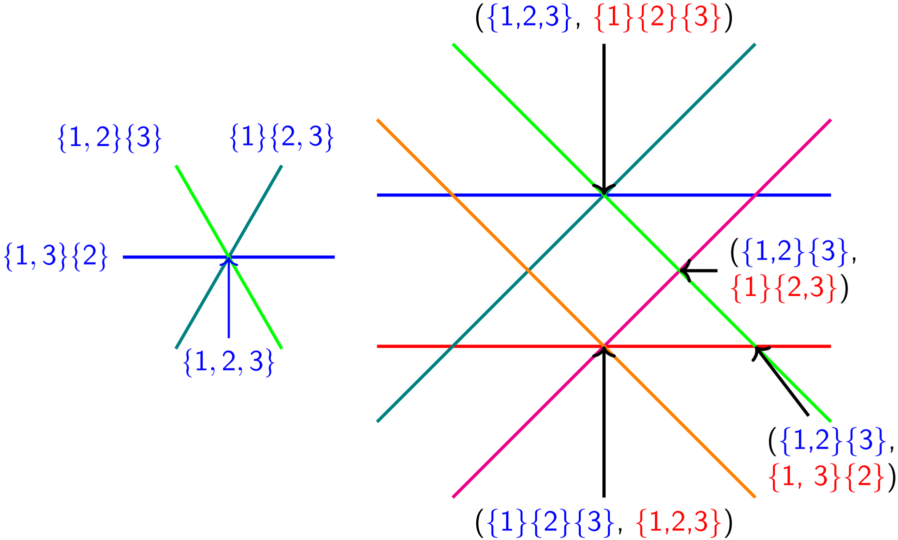

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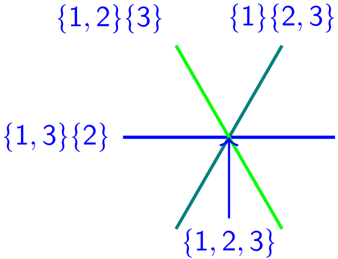

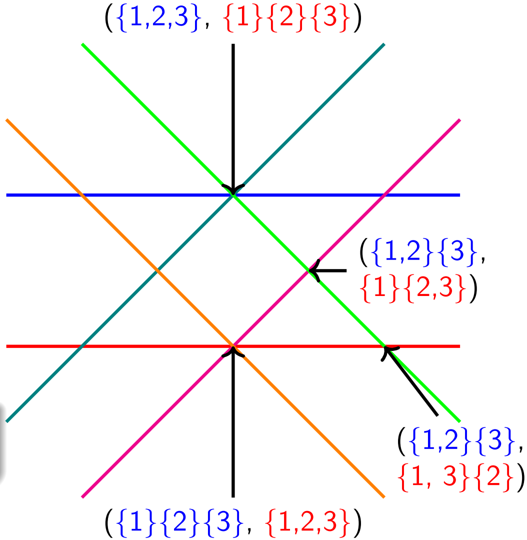

Not every pair is possible

~~({1, 2}{3}, {1, 2}{3})~~

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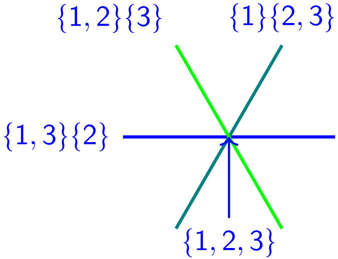

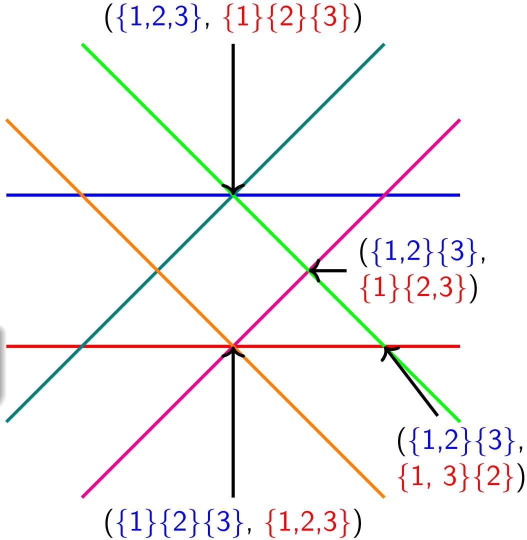

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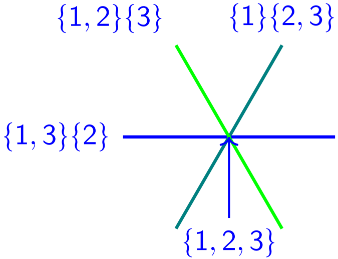
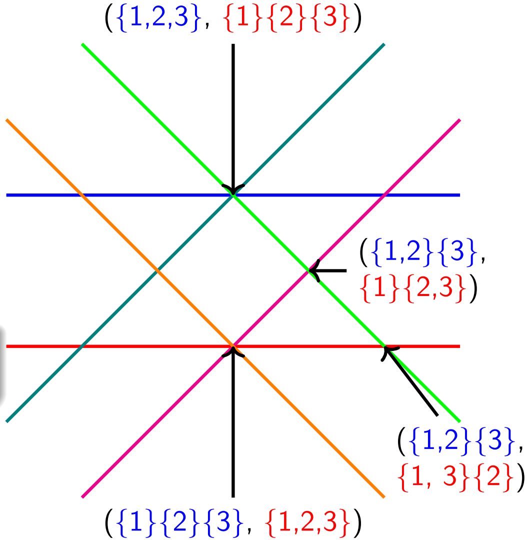
~~({1, 2}{3}, {1, 2}{3})~~

1 2   1

3   23

Description of faces in terms of trees





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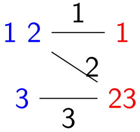

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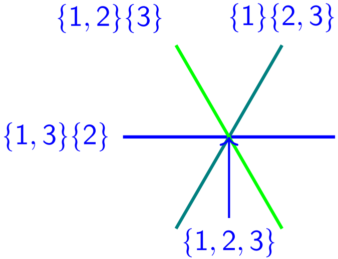

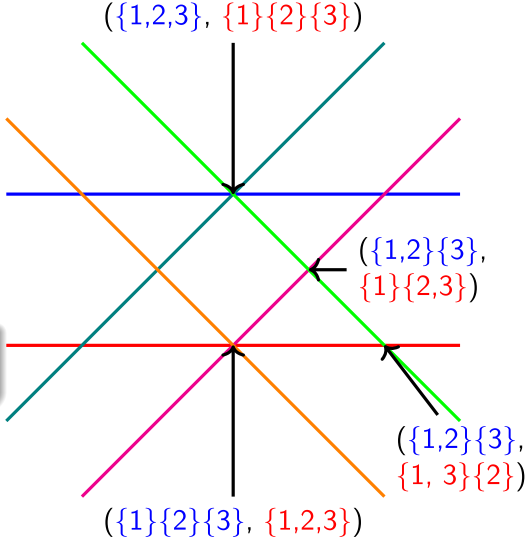

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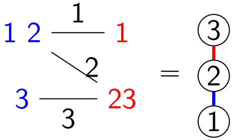
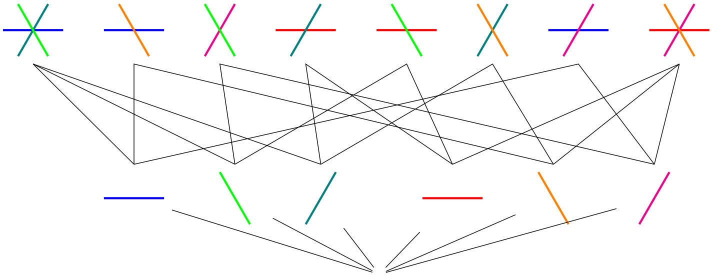


From intersections of hyperplanes to coloured forests

Intersection of hyperplanes

Each intersection is a forest of edge-coloured rooted trees s.t. :

- there are ℓ different colours of edges and 1 is a root
- a child edge does not have the same colour as its parent.



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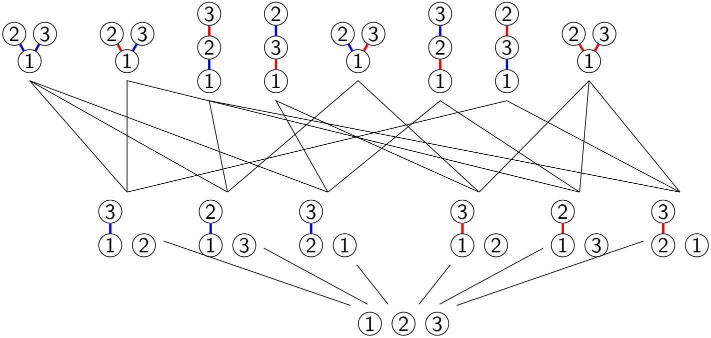

Formula for the number of regions of 2 copies of the braid arrangement

Theorem (BDO, M. Josuat-Vergès, G. Laplante-Anfossi, V. Pilaud, K. Stoeckl)

$$f_{n-k_1-1,n-k_2-1}({\mathcal{B}_n}^2) = \sum_{\mathbf{F}\leqslant\mathbf{G}} \prod_{i\in[2]} \prod_{p\in G_i} (\#F_i[p]-1)!$$

where **F** *and* **G** *are two forests of* 2*-edge-coloured trees and* $\#F_i = k_i + 1$

$$f_{n-1}({\mathcal{B}_n}^2) = (n+1)![x^n]exp\left(\sum_{m\geqslant 1}\frac{x^m}{m(m+1)}\binom{2m}{m}\right)[A213507]$$

$$f_0({\mathcal{B}_n}^2) = 2(n+1)^{n-2}[A007334]$$

which admits the following refinement :

$$f_{k,n-k-1}({\mathcal{B}_n}^2) = \frac{1}{k+1}\binom{n}{k}(k+1)^{n-k-1}(n-k)^k$$

Formula for the number of regions of ℓ copies of the braid arrangement

Theorem (BDO, M. Josuat-Vergès, G. Laplante-Anfossi, V. Pilaud, K. Stoeckl)

$$f_{n-k_1-1,\dots,n-k_\ell-1}(\mathcal{B}_n{}^{\ell}) = \sum_{\mathbf{F}\leqslant\mathbf{G}} \prod_{i\in[\ell]} \prod_{p\in G_i} (\#F_i[p]-1)!$$

where **F** *and* **G** *are two forests of ℓ-edge-coloured trees and $\#F_i = k_i + 1$*

$$f_{n-1}(\mathcal{B}_n{}^{\ell}) = ??$$

$$f_0(\mathcal{B}_n{}^{\ell}) = \ell\,(1+(\ell-1)n)^{n-2}$$

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Merci de votre attention !