# Parking and Tamari-parking posets

Bérénice Delcroix-Oger
joint work with Hélène Han (ENS Saclay), Matthieu Josuat-Vergès (IRIF) et Lucas Randazzo (Nomadic Labs)

Based on : `https://doi.org/10.37236/10714`, Hélène's report and my HDR manuscript

Seminar at Chungbuk National University
Cheongju, April 2025

# Outline

1. From partition posets to parking posets
2. Parking posets (with M. Josuat-Vergès and L. Randazzo)
3. Tamari-parking posets (with M. Josuat-Vergès and H. Han)

# From partition posets to parking posets

# Outline

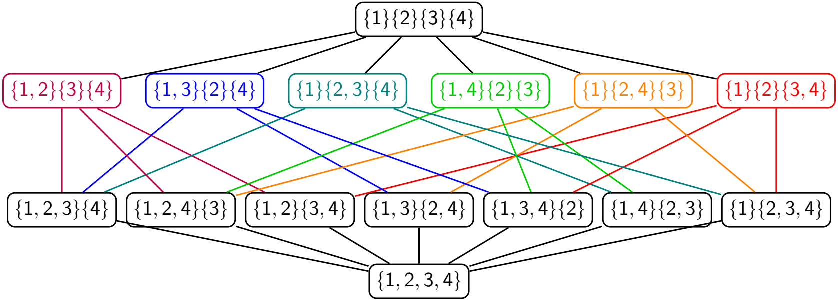
# Posets(=partially ordered set) of (set) partitions $\Pi(V)$

Partitions of a set $V$ :

$$\{V_1, \ldots, V_k\} \models V \Leftrightarrow V = \bigsqcup_{i=1}^{k} V_i,\, V_i \cap V_j = \emptyset \text{ for } i \neq j$$

Partial order on set partitions of a set $V$:

$$\{V_1, \ldots, V_k\} \leqslant \{V'_1, \ldots, V'_p\} \Leftrightarrow \forall i \in \{1, p\}, \exists j \in \{1, k\} \text{ s.t. } V'_i \subseteq V_j$$



# Order complex and cohomology of a poset

The order complex of a bounded poset, i.e. a poset with a minimum and a maximum, is the simplicial complex whose set of faces of size $k-1$ is

$$f_k(P) = \{x_0 < \ldots < x_k \in P | x_0 \in \min(P), x_k \in \max(P)\}.$$

The cohomology of the poset is the cohomology of its order complex.
When $P$ is not bounded, its order complex is the order complex of the smallest bounded poset containing $P$.

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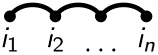
## Results on partition poset of a set of size $n$

- It is the intersection lattice of the braid arrangement
- Its cohomology is $Lie(n) \otimes_{\mathfrak{S}_n} sgn_n$, where $Lie(n)$ is the representation of the symmetric group corresponding to the Lie operad and $sgn_n$ is the signature representation.
- Posets of ordered partitions are the face lattice of the permutohedra

cf. "Cellular diagonals of permutahedra" with G. Laplante-Anfossi, V. Pilaud and K. Stoeckl (ArXiv : 2308.12119)

# Noncrossing partitions [Kreweras, 1972]

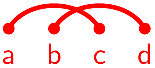
$\{i_1, \ldots, i_n\}$ with $i_1 < \ldots < i_n \rightarrow$ 


## Definition (Kreweras, 1972)

A partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of $\{1, \ldots, n\}$ is noncrossing iff

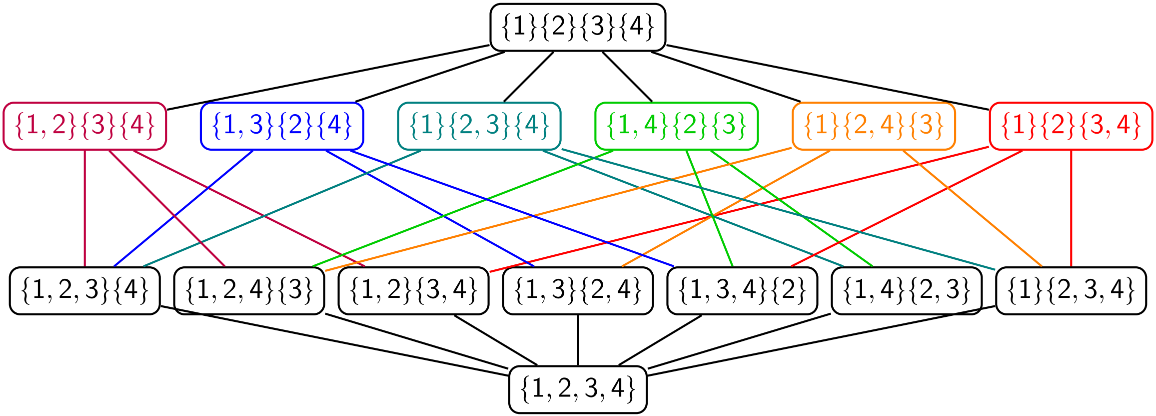
$$\begin{cases} a < b < c < d \\ a, c \in \pi_i \\ b, d \in \pi_j \end{cases} \implies i = j$$

$NC_n$ = set of noncrossing partitions of $\{1, \ldots, n\}$




$\rightarrow$ counted by Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$

# Partition posets



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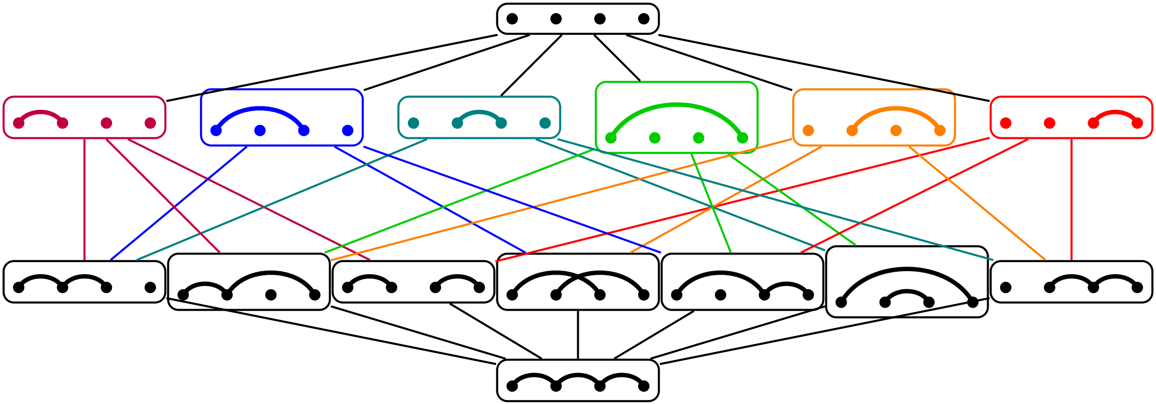

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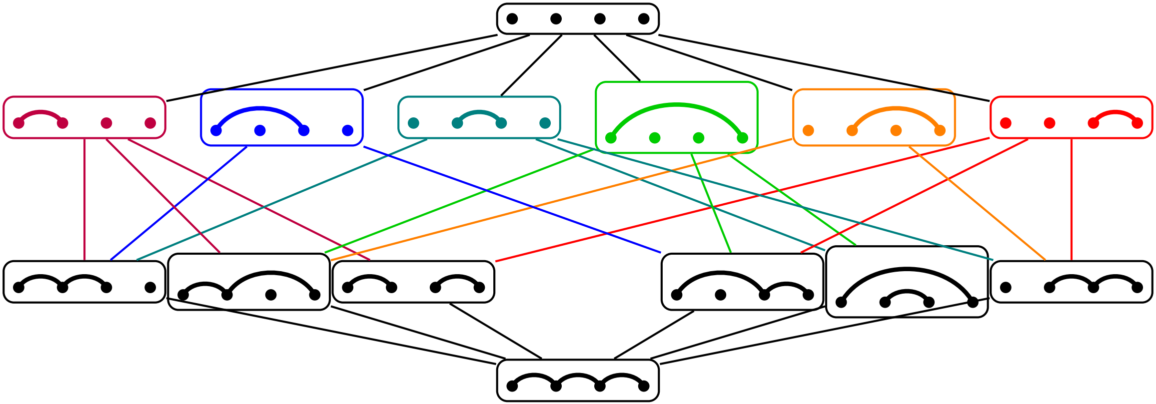

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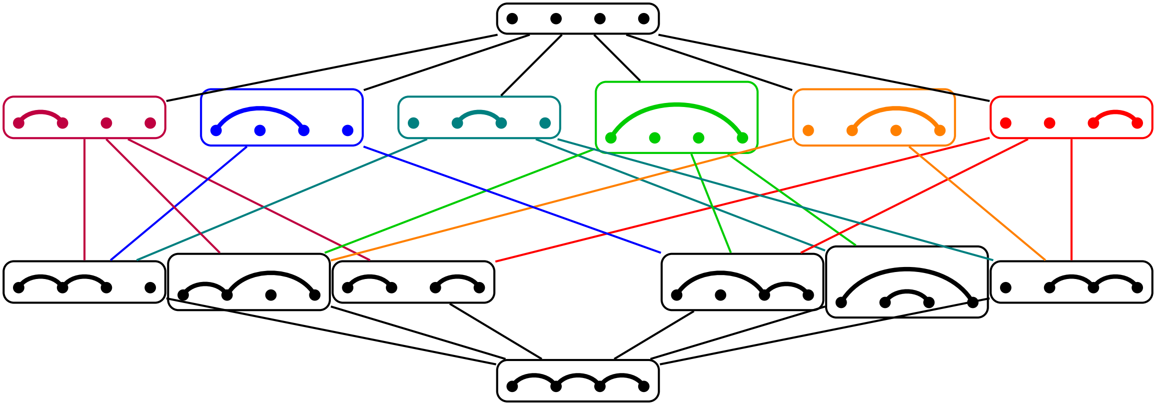


**Problem**

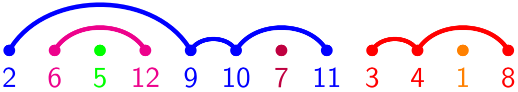
No action of the symmetric group on this poset !

# Noncrossing 2-partitions

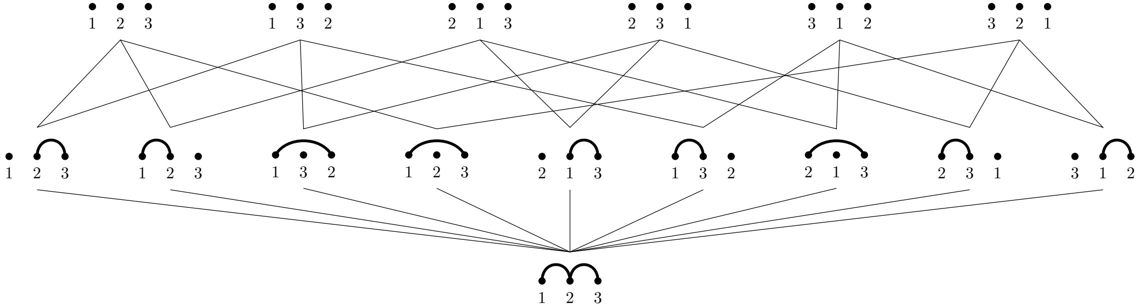
**Definition (Edelman, 1980)**

A n.c. 2-partition of size $n$ is a pair $(\pi, \sigma) \in NCP_n \times \mathfrak{S}_n$ s.t.

$$\begin{cases} \{b_1, \ldots, b_k\} \in \pi \\ b_1 < b_2 < \ldots < b_k \end{cases} \implies \sigma(b_1) < \sigma(b_2) < \ldots < \sigma(b_k).$$



# Noncrossing 2-partition poset



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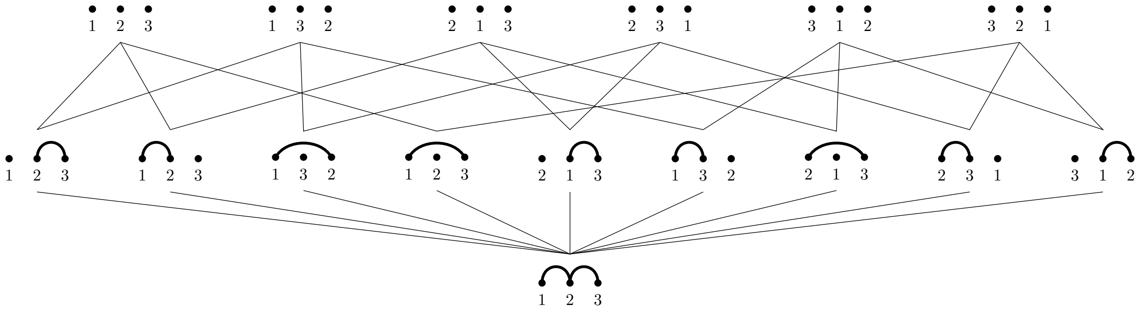


## How many noncrossing 2-partitions are there ?

There are $(n+1)^{n-1}$ noncrossing 2-partitions, which is the same as the number of parking functions on $n$ parking spots.

# Let's drive and park !

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

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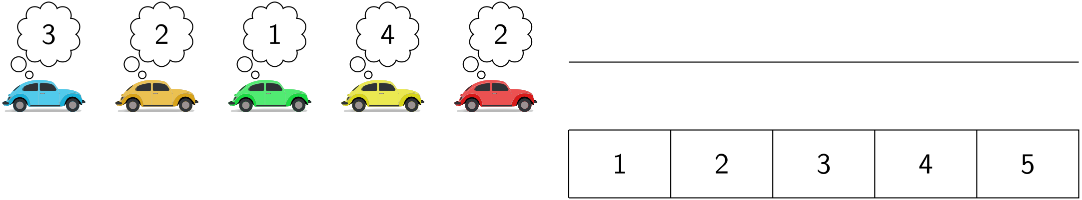

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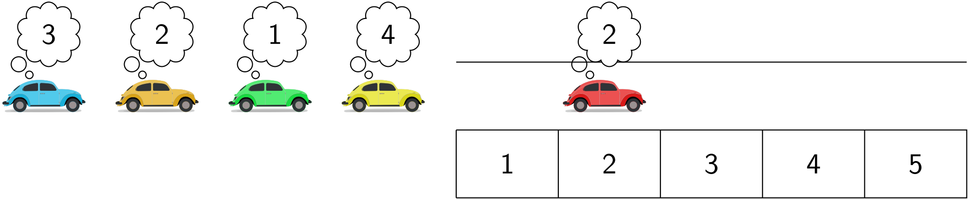

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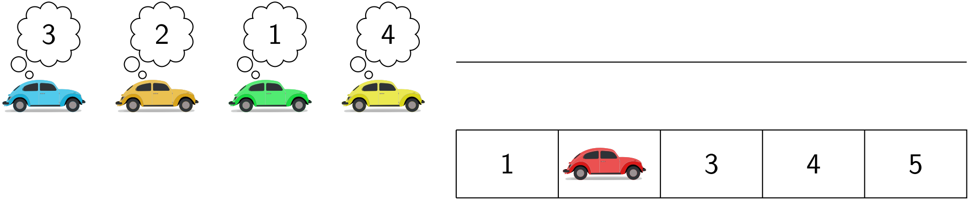

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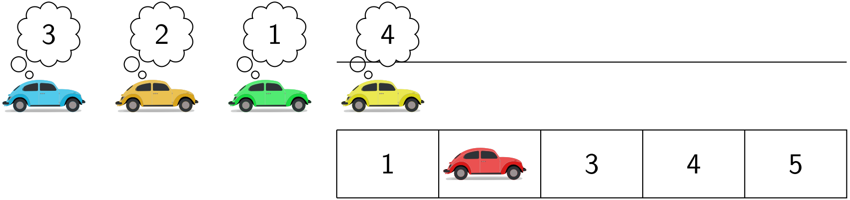

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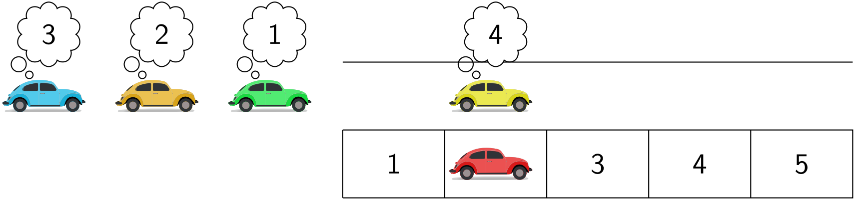

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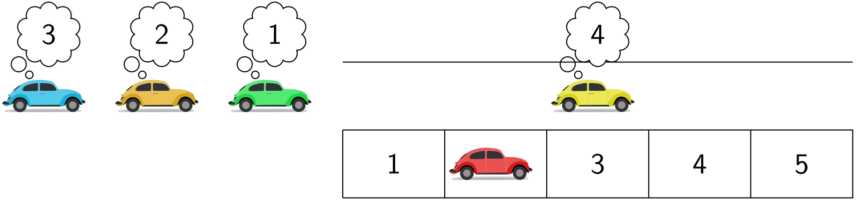

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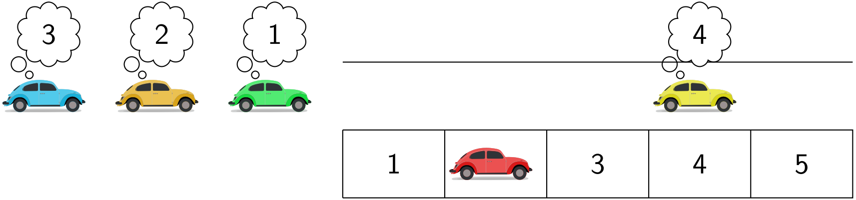

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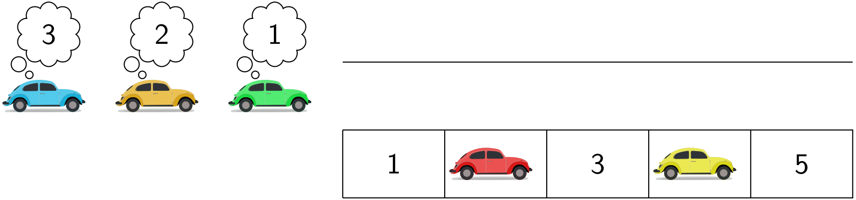

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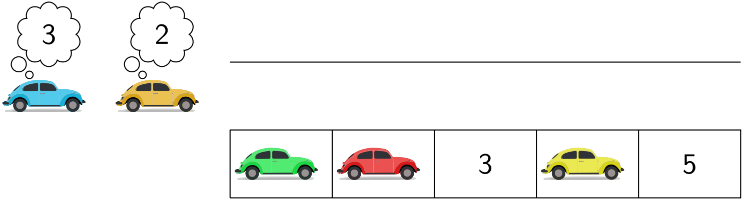

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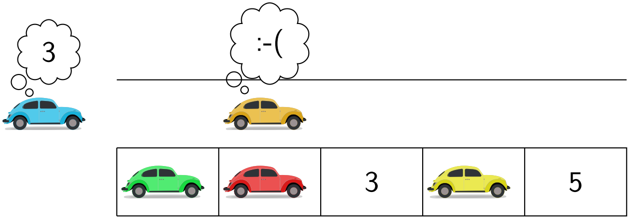

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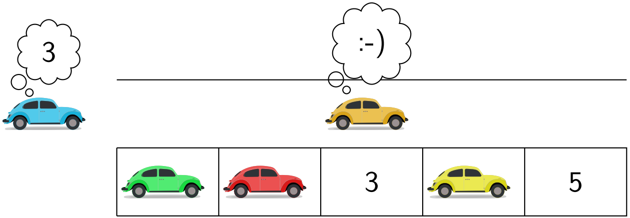

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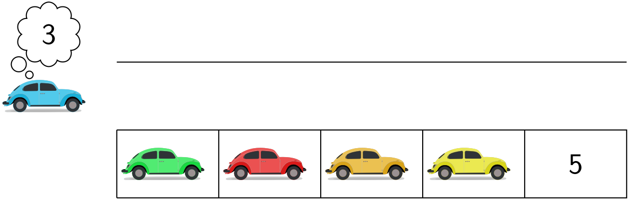

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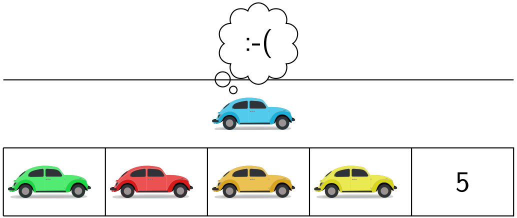

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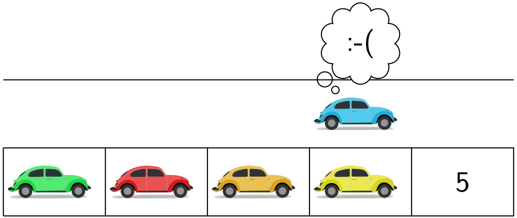

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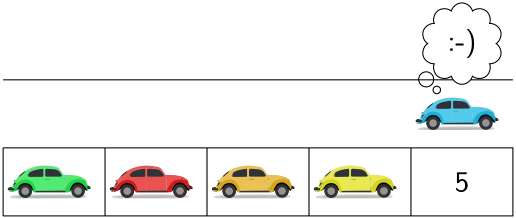

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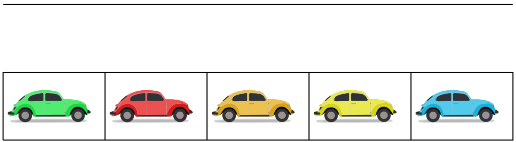

## Definition

If every car can park, the word, obtained by reading from the first car entering the street to the last one, is called a parking function.

## Examples and counter-examples:

| Parking functions of length $n$ | Other words of length $n$ |
|---|---|
| 1 | |
| 11,12,21 | 22 |
| 111, 112, 121, 211, 113, 131, 311,122, 212, 221, 123, 132, 213, 231, 312, 321 | 222, 333, 223, 232, 322, 233, 323, 332, 133, 313, 331 |

# Formal definition of parking functions [Konheim-Weiss, 1966]

## Definition

A sequence $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{N}^*)^n$ is a parking function of length $n$ iff

$$|\{i | 1 \leqslant a_i \leqslant j\}| \geqslant j.$$

Denoting by $a^{\uparrow}$ the non-decreasing rearrangement of $\mathbf{a}$, this is equivalent to $1 \leqslant a_j < j$ for any $1 \leqslant j \leqslant n$. We call non-decreasing parking function a parking function satisfying $a = a^{\uparrow}$.

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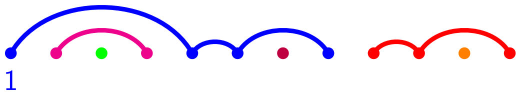
**Theorem (Konheim-Weiss, 1966; Pollak, 1969)**

*There are $(n+1)^{n-1}$ parking functions of length $n$.*
*There are $C_n = \frac{1}{n+1}\binom{2n}{n}$ non-decreasing parking functions of length $n$.*

# Parking functions and noncrossing partitions




Label $i$ by $\min \pi$ for $i \in \pi$

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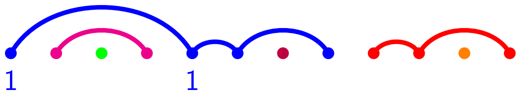


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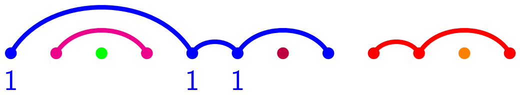


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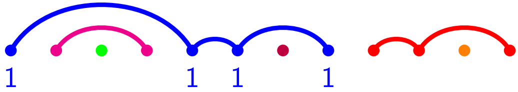


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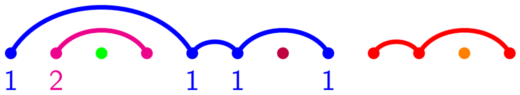


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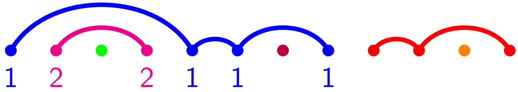


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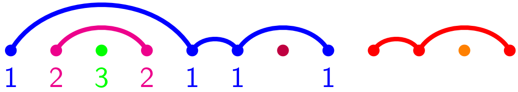


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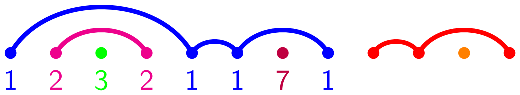


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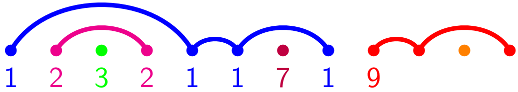


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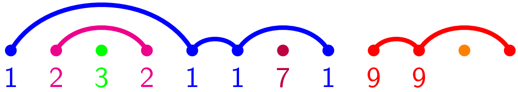


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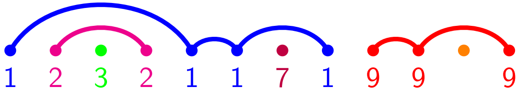


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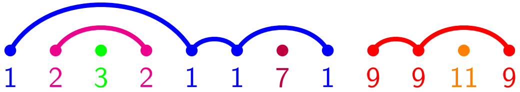


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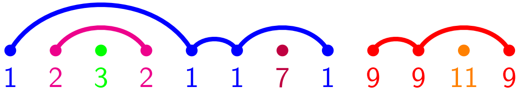


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- Gives a parking function as the label of the $j$th node is smaller or equal to $j$.

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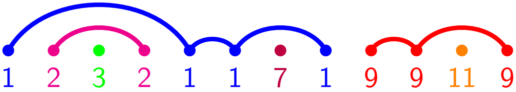
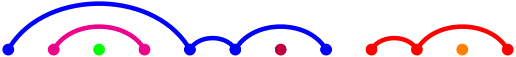



Label $i$ by $\min \pi$ for $i \in \pi$

- Gives a parking function as the label of the $j$th node is smaller or equal to $j$.
- It is the unique parking function in the orbit which maximizes the number of lucky cars. Call it the lucky parking function (used by Blass and Sagan to compute the Möbius function of Tamari lattices under the name "left bracket vector").

# Posets of noncrossing 2-partitions in terms of parking functions



1 2 3 2 1 1 7 1 9 9 11 9

2 6 5 12 9 10 7 11 3 4 1 8

in terms of parking function:

- - - - - - - - - - - -

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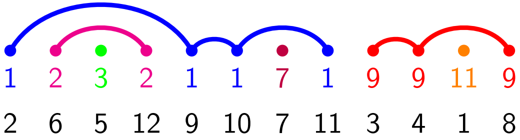


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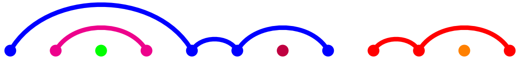

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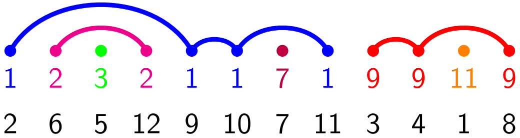


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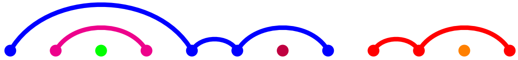

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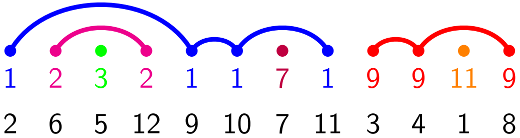


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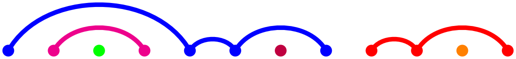

1 2 3 2 1 1 7 1 9 9 11 9

2 6 5 12 9 10 7 11 3 4 1 8

in terms of parking function:

11 1 9 9 3 2 7 9 1 1 1 2

# Parking functions also appear

- as labellings of the shi arrangement
- as labellings of maximal chains in the noncrossing partition poset
- in two posets:
  - the poset of 2-noncrossing partitions [Edelman, 80]
  - the poset of Tamari-parking, linked with the study of diagonal coinvariants [Chapuy–Bousquet-Mélou–Préville-Ratelle, 13]

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- as labellings of the shi arrangement
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**Our goal for today:**

- in two posets:
  - the poset of 2-noncrossing partitions [Edelman, 80]
  - the poset of Tamari-parking, linked with the study of diagonal coinvariants [Chapuy–Bousquet-Mélou–Préville-Ratelle, 13]

# Parking posets (with M. Josuat-Vergès and L. Randazzo)

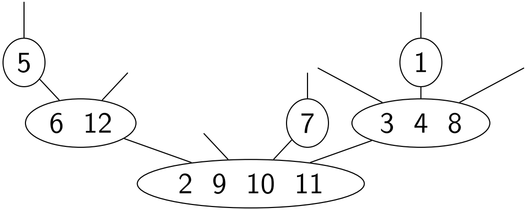
# Outline

# Parking trees

## Definition

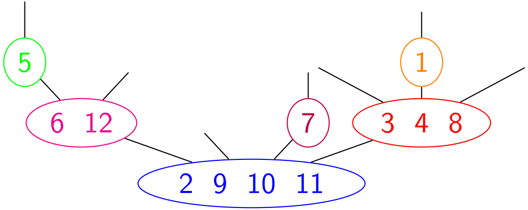
A parking tree on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

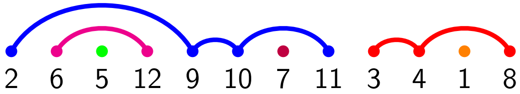
- $V \in \Pi_L$,
- $v \in V$ has $|v|$ children.



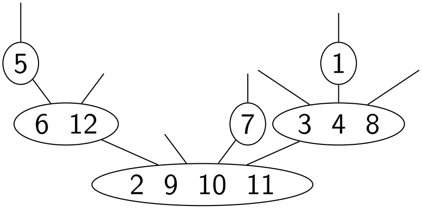

Why parking ?

# Bijection between 2-noncrossing partition and parking trees



# Functional equation for parking trees




**Proposition (DO, Josuat-Vergès, Randazzo, 22)**

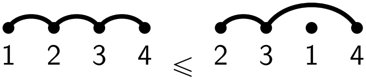
$$\mathcal{P}_f = \sum_{p \geqslant 1} \mathbb{E}_p \times (1 + \mathcal{P}_f)^p$$

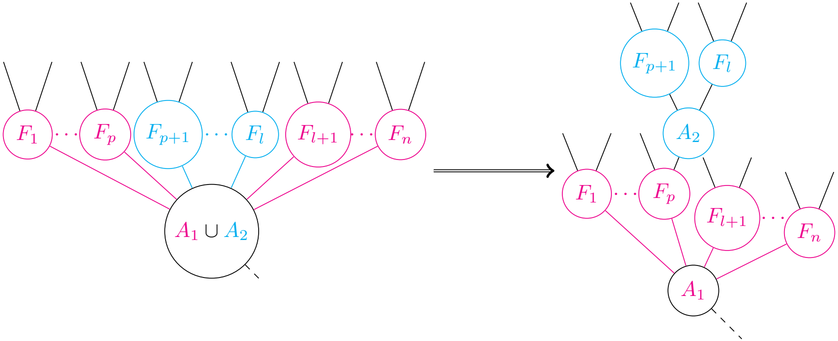
$$\mathcal{P}_f = \sum_{p \geqslant 1} \frac{x^p (1 + \mathcal{P}_f)^p}{p!} = \exp\left(x(1 + \mathcal{P}_f)\right)$$

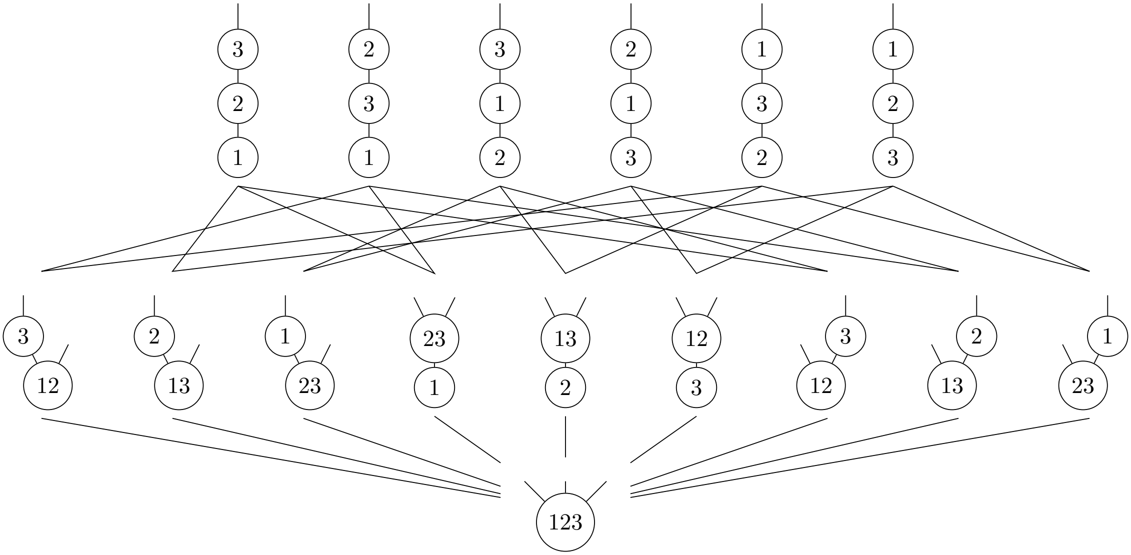
# Noncrossing 2-partitions poset

Covering relation in $\Pi^2$ : merge parts and rearrange labels to respect the increasing condition

Example :

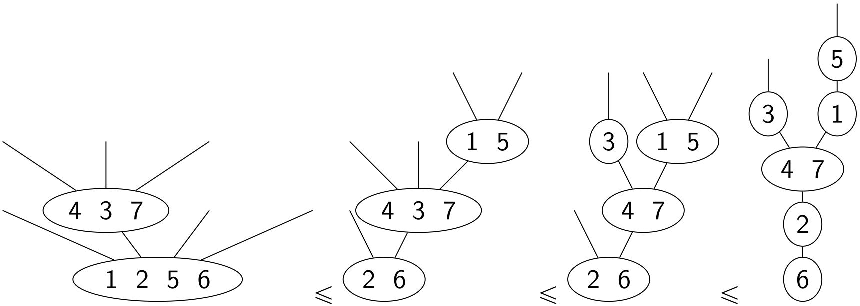


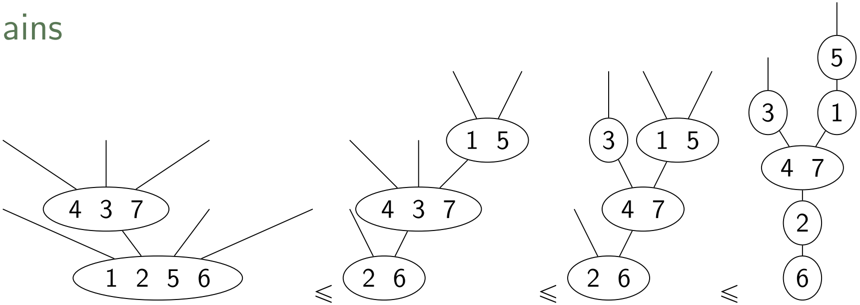

3 2 3 2 1 1
2 3 1 1 3 2
1 1 2 3 2 3
3 2 1 23 13 12 3 2 1
12 13 23 1 2 3 12 13 23
123

# Results

- This poset is a lattice
- When restricting to right combs, get the face poset of the permutohedron
- New criterion to prove shellability !
- Enumeration of (weak) k-chains, hence computation of Euler characteristic



# $k$-weak chains




## Proposition (DO, Josuat-Vergès, Randazzo, 22)

$$\mathcal{C}^{l}_{k,t} = \sum_{p\geqslant 1} \mathcal{C}^{l,p}_{k-1,t} \times \left(t\mathcal{C}^{l}_{k,t} + 1\right)^{p}$$

*Chains $\phi_1 \leqslant \cdots \leqslant \phi_k$ in ${\Pi^2}_n$ are in bijection with $k$-parking trees.*
*The number of chains $\phi_1 \leqslant \cdots \leqslant \phi_k$ in ${\Pi^2}_n$ where $\mathrm{rk}(\phi_k) = \ell$ is:*

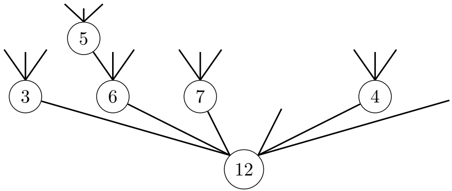
$$\ell! \binom{kn}{\ell} S_2(n, \ell+1).$$

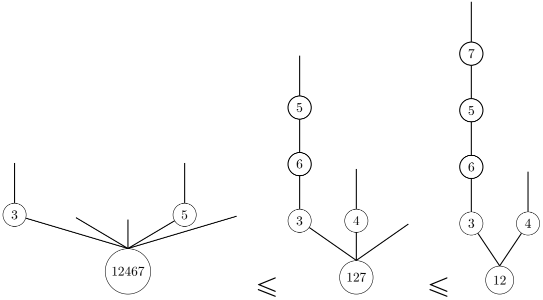
# $k$-parking tree

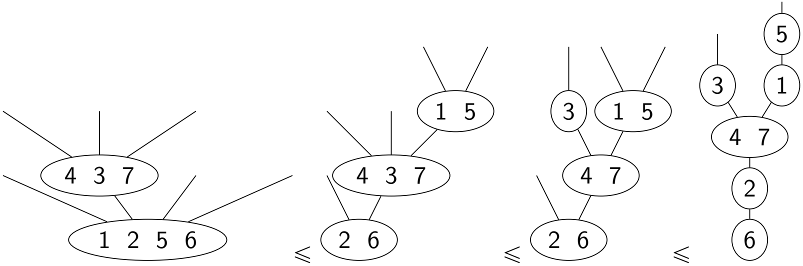
## Definition

A *$k$-parking tree* on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

- $V \in \Pi_L$
- $v \in V$ has $k|v|$ children.




4 3 7
1 2 5 6
⩽
1 5
4 3 7
2 6
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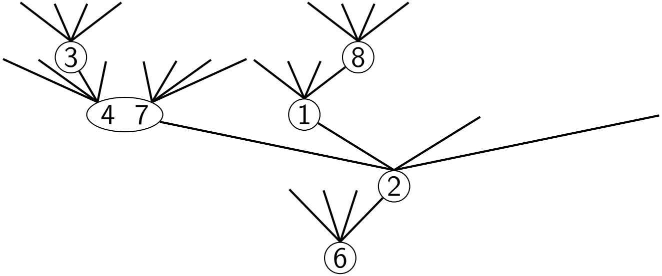
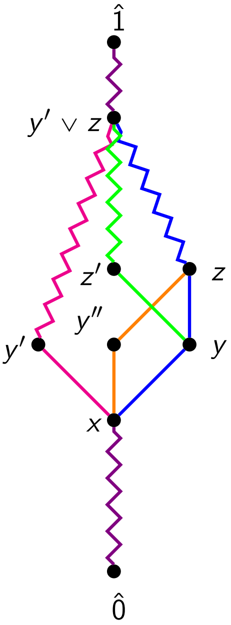
3
1 5
4 7
2 6
⩽
5
3
1
4 7
2
6


3
8
4 7
1
2
6

# Shelling




## Lemma (D.O., Josuat-Vergès, Randazzo, 22)

*Let* $x, y, y', z \in {\Pi^2}_n$ *such that* $x \lessdot y \lessdot z$, $x \lessdot y'$, *and* $y' \prec_x y$. *Then:*

- *either there exists* $y'' \in {\Pi^2}_n$ *such that* $x \lessdot y'' \lessdot z$ *and* $y'' \prec_x y$,
- *or there exists* $z' \in {\Pi^2}_n$ *such that* $y \lessdot z' \leqslant y' \vee z$ *and* $z' \prec_y z$.

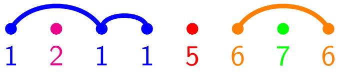
# Tamari-parking posets (with M. Josuat-Vergès and H. Han)

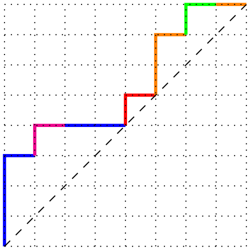
# Outline

# Another Catalan object : Dyck paths

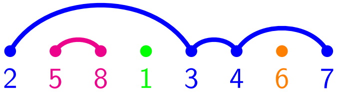
## Definition

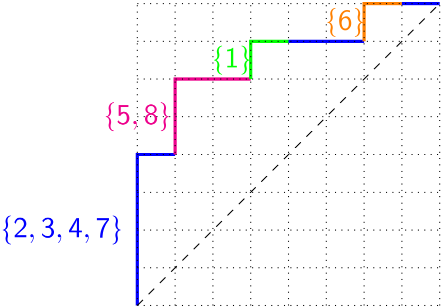
A Dyck path of size $n$ is a path in $\mathbb{Z}^2$ from $(0,0)$ to $(n,n)$ using exactly $n$ north steps and $n$ east step.

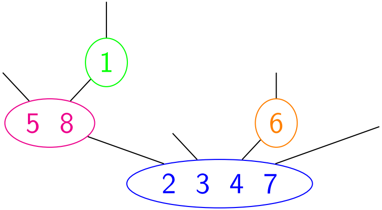


# Noncrossing 2-partitions, labelled Dyck path and parking functions



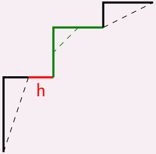



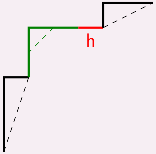

41112712

# Tamari lattices on noncrossing partitions

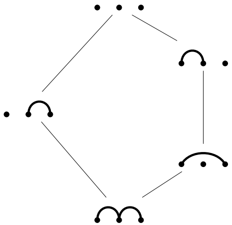
Tamari lattices were first introduced by Tamari in 1951 in terms of rotations of a planar binary tree. We give here an analogous definition in terms of Dyck path and noncrossing partitions.

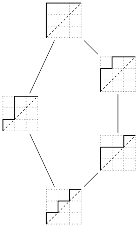
## Definition

$C =$ 

$\lessdot$ 

$= D$



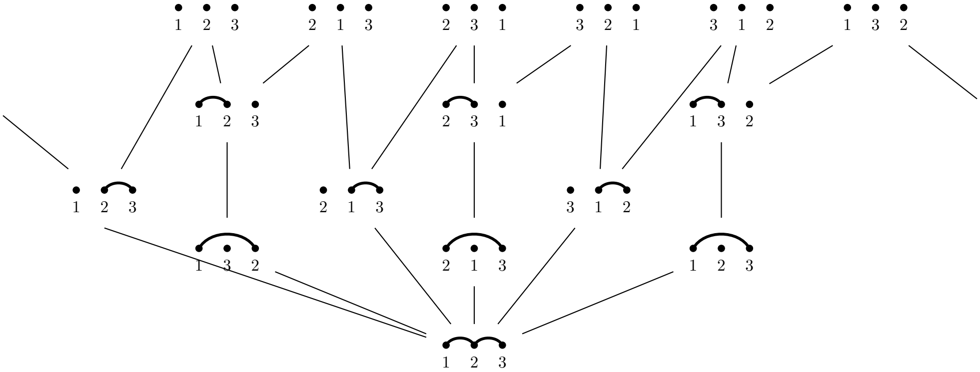


# Tamari-parking poset

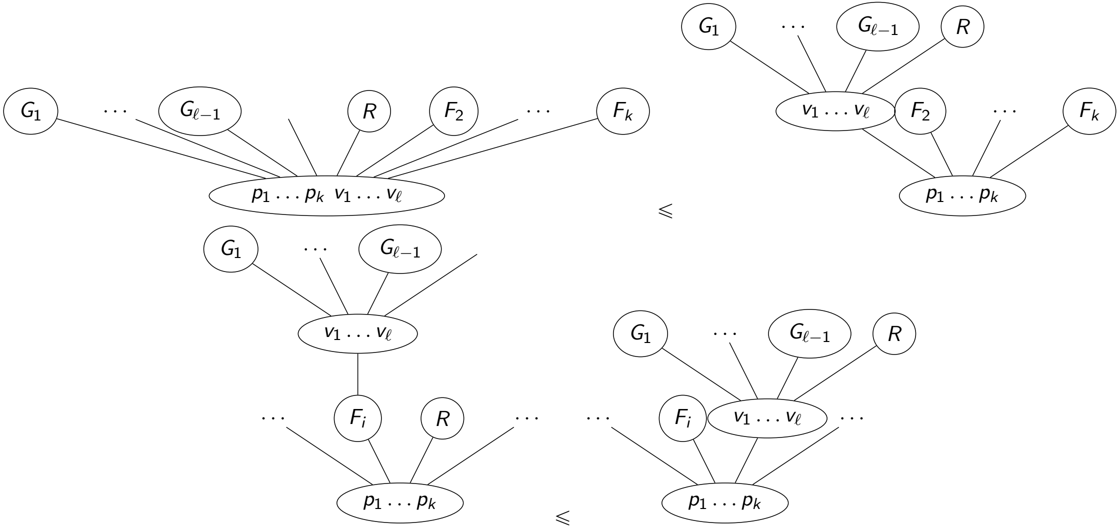
Covering relations given by:

- Moving a block to an arch to the left in the same part 

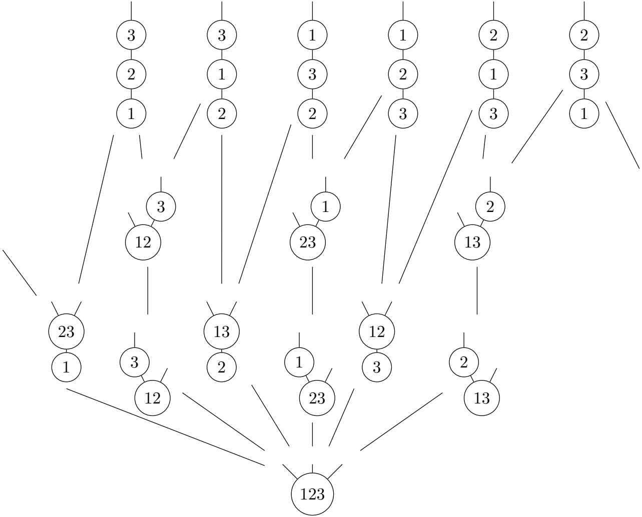
- Merging parts if their leftmost elements are adjacents 

# Covering relations on Tamari-Parking posets



# Tamari-parking poset



# Known results and open questions

**Proposition (Chapuy–Bousquet-Mélou–Préville-Ratelle, 12)**

*Say that two intervals in Tamari-parking posets are isomorphic if they have the same minimum element and maximal elements of the same shape.*
*The number of class of isomorphisms of intervals is given by:*

$$2^n(n+1)^{n-2}.$$

The action of the symmetric group on these isomorphisms class of intervals are likely to be the same as the one on the space of diagonal coinvariants in three sets of $n$ variables.

**Conjecture (DO)**

Augmented Tamari-parking posets are homotopic to a sphere.

**Proposition (H. Han)**

*Tamari-parking posets are lattices. They are neither EL-shellable nor CL-shellable.*

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*Tamari-parking posets are lattices. They are neither EL-shellable nor CL-shellable.*

Thank you for your attention !

# Parking functions and Cayley trees

- $(n+1)^{n-1}$ is also the number of Cayley trees on $n+1$ vertices (or equivalently of forest of rooted Cayley trees on $n$ vertices)
- There are several bijections between these objects (see Yan's survey for instance) which enable to refine the enumeration of parking functions with statistics such as displacements, number of lucky cars, ...

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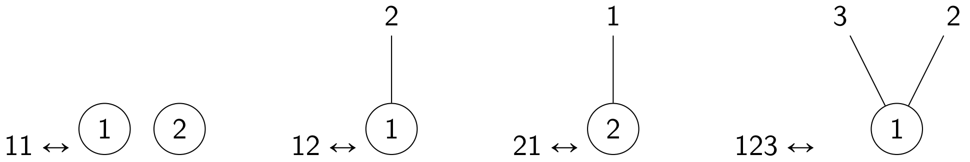
## Pollak's bijection

Consider the Cayley tree associated with the Prüfer code $(c_1, \ldots, c_{n-1})$ where

$$c_i \equiv a_{i+1} - a_i [n+1]$$



# Action of the symmetric group on parking functions and Cayley forests

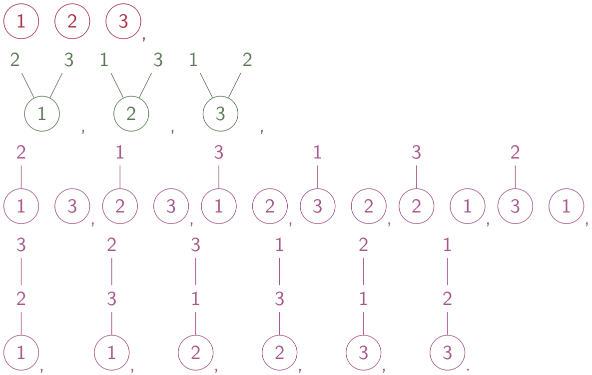
111,

112, 121, 211,

113, 131, 311,

122, 212, 221,

123, 132, 213, 231, 312, 321.



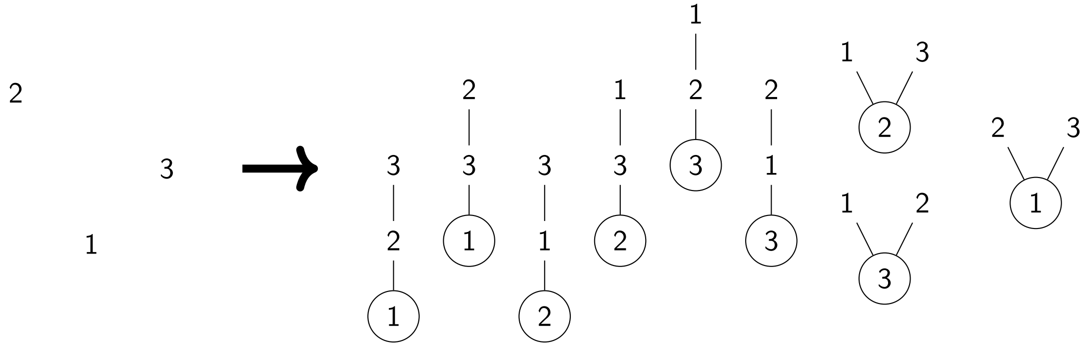
# What are species?

**Definition (Joyal, 80s; cited from Bergeron-Labelle-Leroux)**

A species F is a functor from Bij to Set. To a finite set $S$, the species F associates a finite set F($S$) such that any bijection $\sigma : S \to T$ gives rise to a map $F(\sigma) : F(S) \to F(T)$ satisfying

$$\forall \sigma : S \to T, \tau : T \to U, F(\tau \circ \sigma) = F(\tau) \circ F(\sigma), \qquad F(Id_S) = Id_{F(S)}.$$

Species = Construction plan, such that the obtained set is invariant by relabelling



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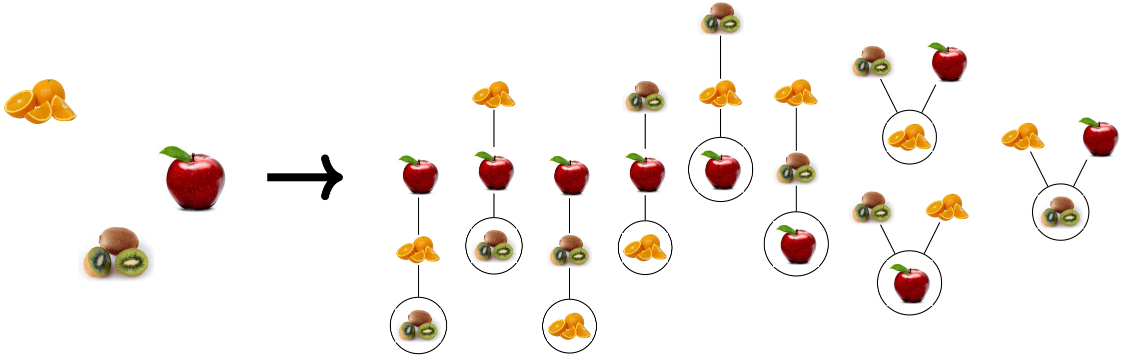
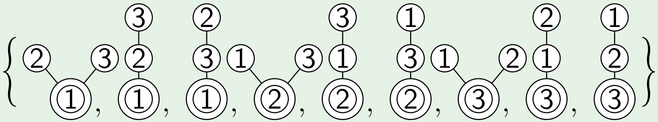
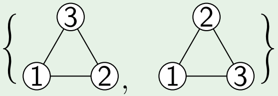


## Examples of species

- $\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists $\mathbb{L}$ on $\{1,2,3\}$)
- $\{\{1,2,3\}\}$ (Species of non-empty sets $\mathbb{E}^+$)
- $\{\{1\},\{2\},\{3\}\}$ (Species of pointed sets $\mathbb{E}^\bullet$)
-  (Species of Cayley trees $\mathbb{T}$)
-  (Species of cycles)

These sets are the image by species of the set $\{1,2,3\}$.

## Why do we need species ?

Let $F$ and $G$ be two species.

- $(F + G)(I) = F(I) \sqcup G(I)$,

- $(F \times G)(I) = \bigsqcup_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$.

# Cycle index series

## Definition

Given a finite set $V$ of size $n$, the cycle type of a permutation $\sigma \in \mathfrak{S}_V$ is the tuple $(\sigma_1, \ldots, \sigma_n)$, where $\sigma_k$ is the number of cycles of type $k$ in the decomposition of $\sigma$ into disjoint cycles.

## Examples

The cycle type of $(123)(4)(567)$ is $(1, 0, 2, 0, 0, 0, 0)$.

## Definition

The cycle index series of a species $F$ is the formal power series

$$Z_F(p_1, \ldots, p_n, \ldots) = \sum_{n \geqslant 0} \sum_{\sigma} \operatorname{fix} F(\sigma) \frac{p_\sigma}{z_\sigma}, \tag{1}$$

where the sum runs over a set of representatives of each cycle type of $\mathfrak{S}_n$, $p_\sigma = p_1^{\sigma_1} \ldots p_n^{\sigma_n}$ and $z_\sigma = \prod_{i \geqslant 1} i^{p_i} \times p_i!$

# Cycle index series of usual species

## Definition

The cycle index series of a species $F$ is the formal power series

$$Z_F(p_1, \ldots, p_n, \ldots) = \sum_{n \geqslant 0} \sum_{\sigma} \mathrm{fix}\, F(\sigma) \frac{p_\sigma}{z_\sigma}, \tag{2}$$

where the sum runs over a set of representatives of each cycle type of $\mathfrak{S}_n$, $p_\sigma = p_1^{\sigma_1} \ldots p_n^{\sigma_n}$ and $z_\sigma = \prod_{i \geqslant 1} i^{p_i} \times p_i!$

## Examples

$Z_{\mathbb{L}} = \frac{1}{1-p_1}$,
$Z_{\mathbb{E}} = \exp(\sum_{i \geqslant 1} \frac{p_i}{i})$
$Z_{\mathbb{E}^\bullet} = p_1 \exp(\sum_{i \geqslant 1} \frac{p_i}{i})$
$Z_{\mathbb{T}} = p_1 \exp(Z_{\mathbb{T}})$