Generalised flip order

Bérénice Delcroix-Oger
with Jovana Obradović (Serbian Academy of Science)
and Pierre-Louis Curien (CNRS-IRIF, Université Paris Cité)

Workshop "Algebraic Combinatorics and Finite Groups IV"
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Outline

1. Weak order, Tamari order and their extensions to the faces of the permutohedron and associahedron
2. Hypergraph polytopes (aka. nestohedra)
3. From Barnard-McConville's flip order to the Generalised flip order
4. Link with the shuffle products

Weak order, Tamari order and their extensions to the faces of the permutohedron and associahedron

Outline

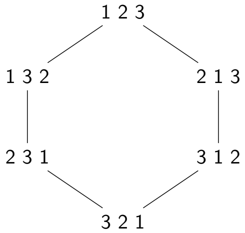
Weak Bruhat order [Verma 1968]

On permutations $\sigma \in \mathfrak{S}_n$ represented as $\sigma(1)\dots\sigma(n)$

1234

Covering relations :

$$\dots i+1 \dots i \dots \lhd \dots i \dots i+1 \dots$$



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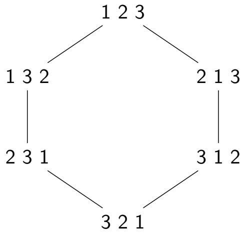

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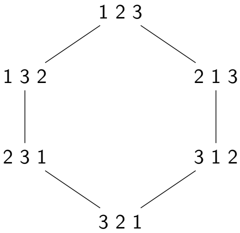

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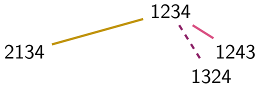

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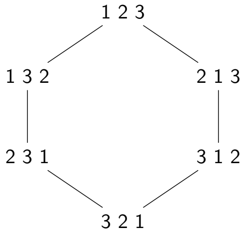

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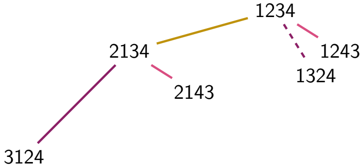

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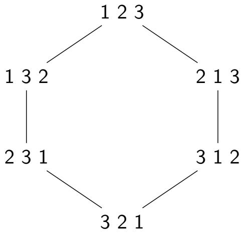

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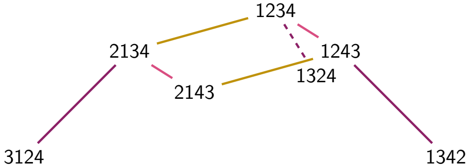

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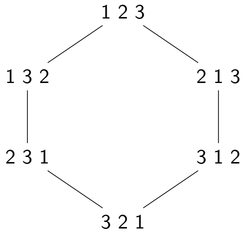

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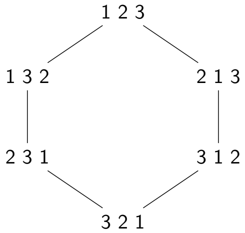
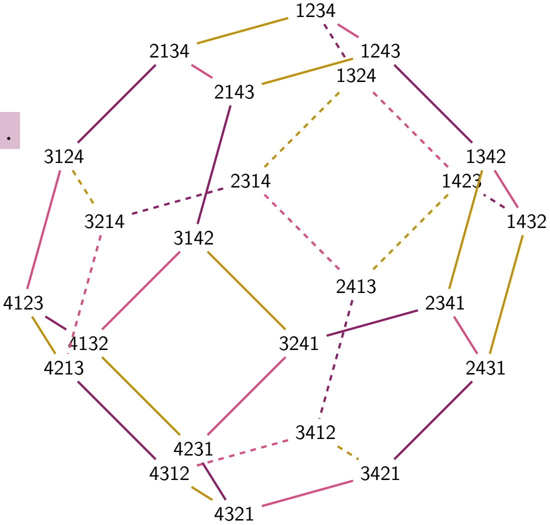
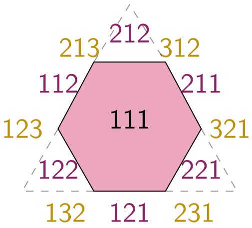


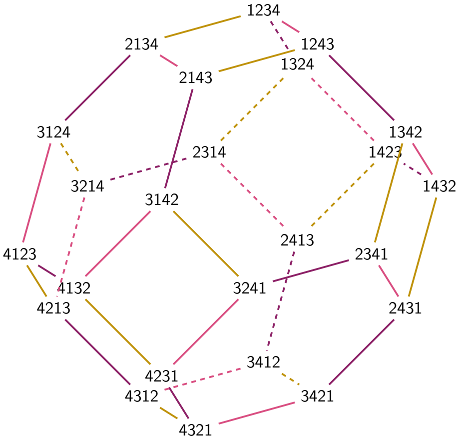


Permutohedra [Schoute 1911]

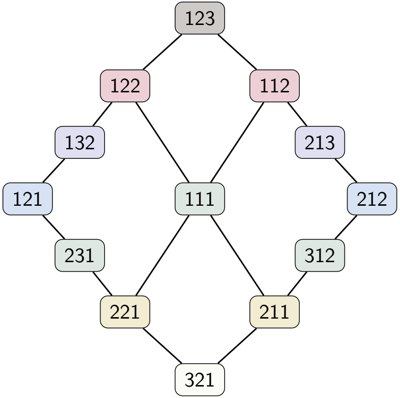
vertices=permutation of $[\![1;n]\!]$

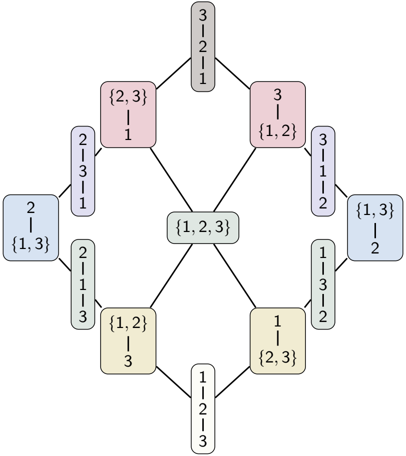
k-dimensional face = a surjection from $[\![1;n]\!]$ to $[\![1;k]\!]$ (also know as packed word)



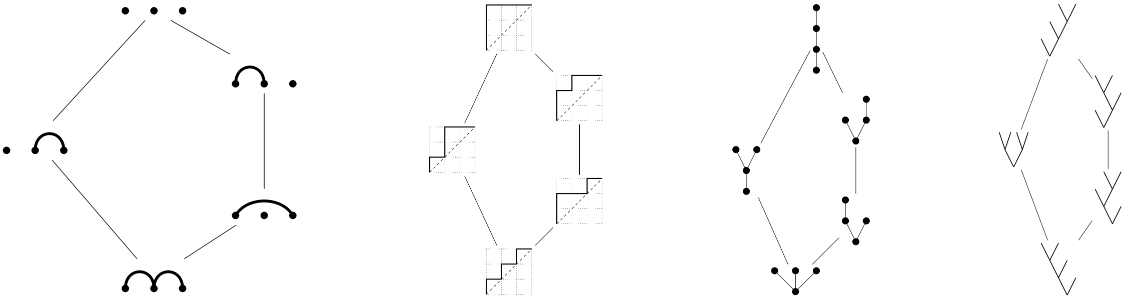


Generalised weak order [Krob-Lapaty-Novelli-Phan-Schwer, 01; Palacios-Ronco, 06; Dermenjian-Hohlweg-Pilaud, 18]





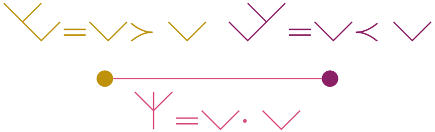
Tamari order

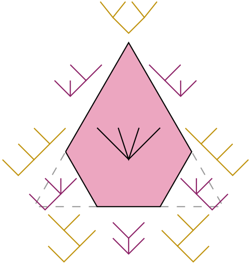


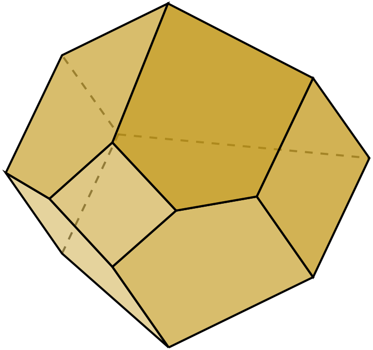
Associated polytope: Associahedra

vertices = planar binary trees

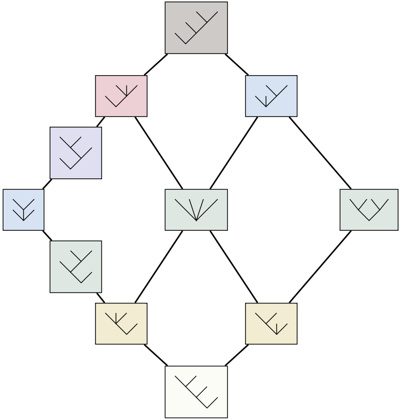
k-dimensional face = planar trees on $n-k$ nodes

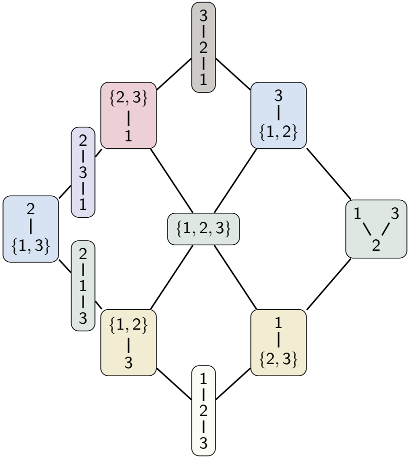






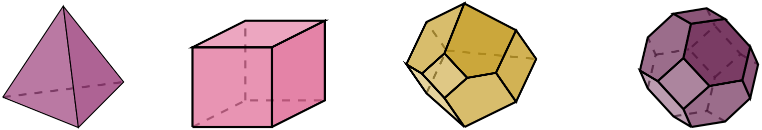
Generalised Tamari order [Ronco, 12]





Hypergraph polytopes (aka. nestohedra)

Outline



Simplices Hypercubes Associahedra Permutohedra

These four polytopes are instances of nestohedra / hypergraph polytopes

Could we define an order on their faces ?

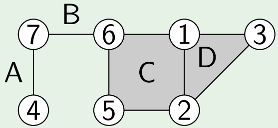
Hypergraphs

Definition

A hypergraph (on vertex set V) is a pair (V, E) where:

- V is a finite set, (the vertex set)
- E is a set of sets of size at least 2, $E \subset \mathcal{P}(V)$.

Example of an hypergraph on $[1;7]$



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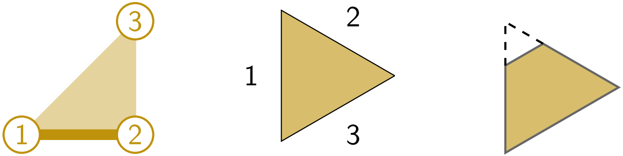
Warning:

All the hypergraphs considered in this talk are connected !

Hypergraph polytope [Došen, Petrić] (=nestohedra [Postnikov])



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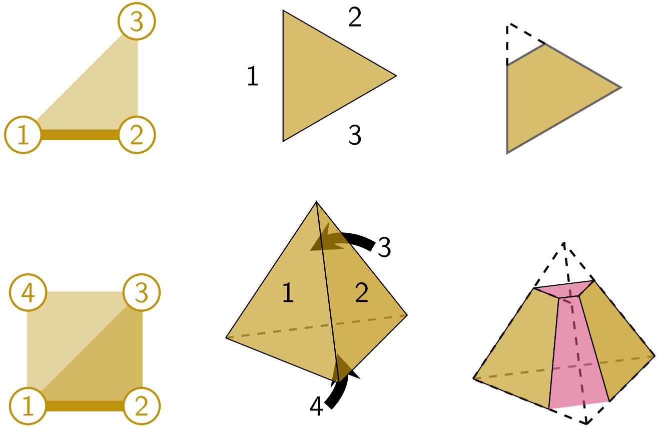


Constructs [Postnikov; Curien-Ivanovic-Obradović]

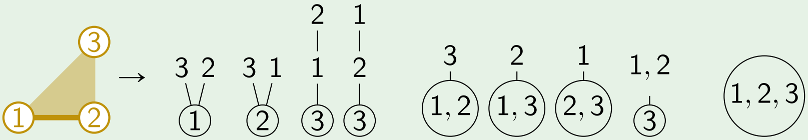
Constructs

A construct c of a hypergraph H is a rooted tree whose vertices form a partition of $V(H)$ and defined inductively by:

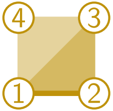
- c has only one node labelled by $V(H)$,
- or the root of c is $E \subseteq V(H)$
  and each of its children is a construct of a connected component of $H - E$.

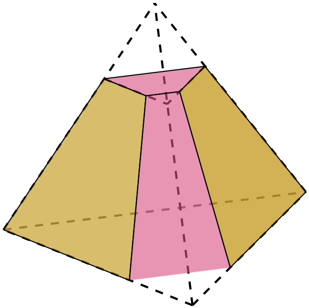
The set of constructs of a given hypergraph labels faces of the associated polytope.

First example:

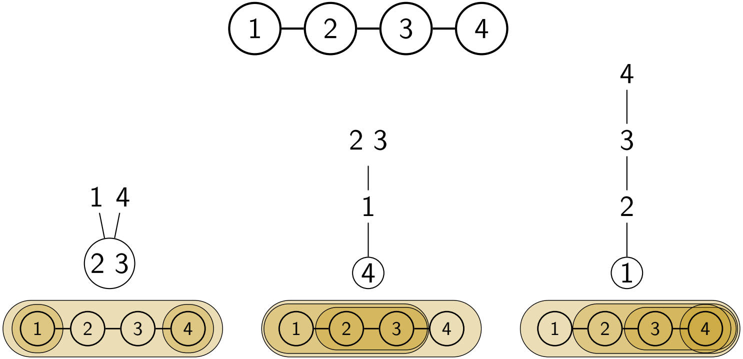


Second example geometrically

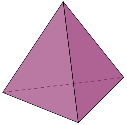
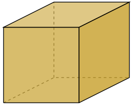
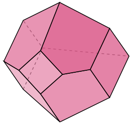
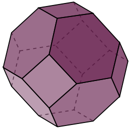
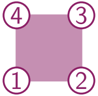
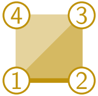
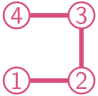
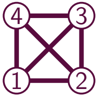




Correspondence Tubings = Constructs = Spines



Upper ideals in constructs = tubes

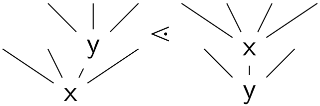
Polytope	Simplex	Hypercube	Associahedron	Permutohedron
Photo				
Associated hypergraph				
Combinatorial objects	multipointed sets	left-comb tree	planar trees	packed words
Cardinality	$2^{n+1} - 1$ (A074909)	3^n (A013609)	Super-Catalan (A001003)	Fubini nbrs (A000670)

From Barnard-McConville's flip order to the Generalised flip order

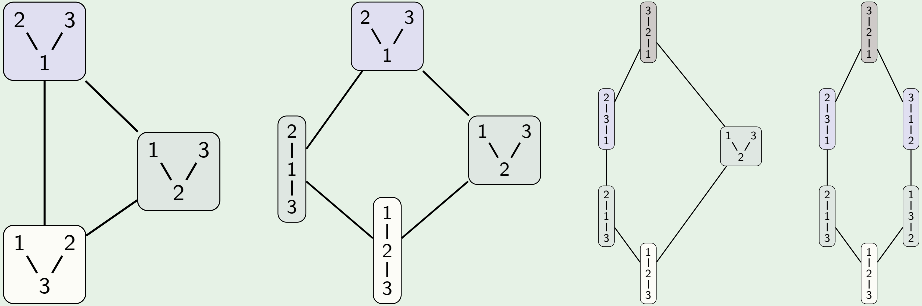
Outline

Barnard-McConville's flip order

For $x < y$,



Examples



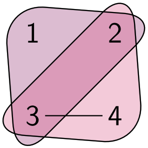
Order on faces of nestohedra : generalised flip order [Curien–Laplante-Anfossi, Curien–D.O.–Obradovic]

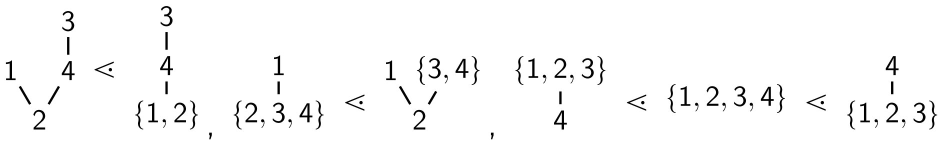
- Hypergraphs on $\mathbb{Z}$ (called **ordered** hypergraph)

Definition

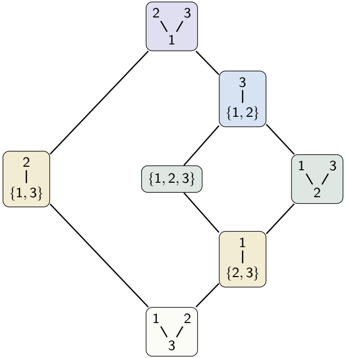
S and T two constructs of $\mathbf{H}$. $S \lessdot T$ if and only if there exists

- U parent of V in S such that $\min(U) > \max(V)$ and T obtained from S by merging U and V, (contraction)
- or V parent of U in T such that $\min(U) > \max(V)$, and S obtained from T by merging U and V. (split)

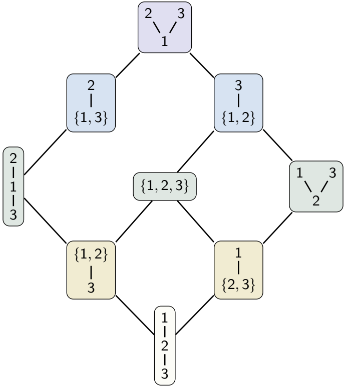




Examples

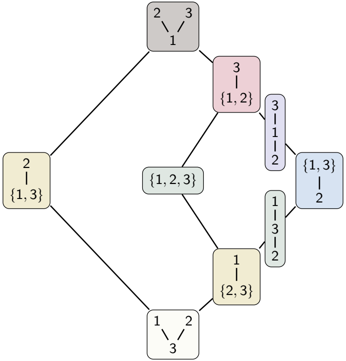


Simplex 1 2 3

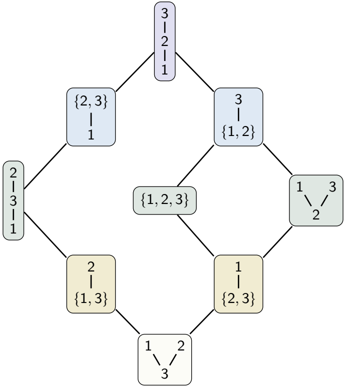


Cube 1 — 2 3

Examples

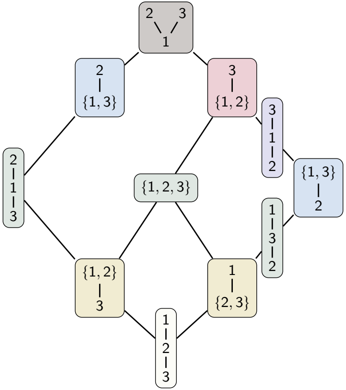


Cube 1 2 3

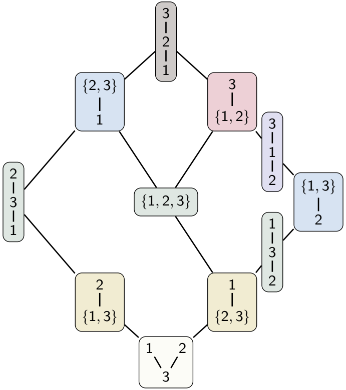


Cube 1 2 — 3

Examples

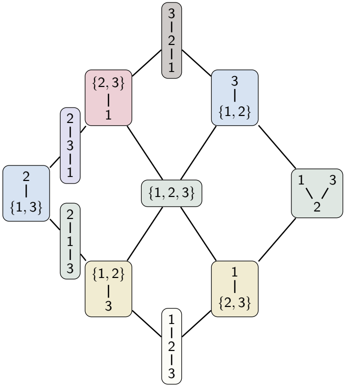


Pentagon $2 — 1 — 3$

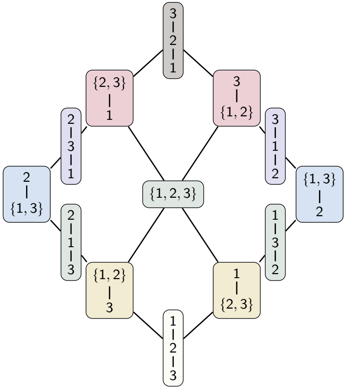


Pentagon $1 — 3 — 2$

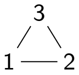
Examples



Pentagon $1 — 2 — 3$



Hexagon



Link between orders

Definition

H is **hereditarily ordered** if for any non-empty $X \subseteq H$, the induced decomposition $\mathbf{H}, X \rightsquigarrow H_1, \ldots, H_n$ can be ordered in such a way that $H_1 < H_2 < \cdots < H_n$.

Proposition (Curien –DO–Obradovic, 26+)

Let $S, T : \mathbf{H}$ be two constructions. The following two properties hold:

1. *If $S \lessdot_{BM} T$, then $S \lessdot^{+} T$.*
2. *If $\mathbf{H}$ is hereditarily ordered and if $S \lessdot^{+} T$, then $S \lessdot^{+}_{BM} T$.*

In terms of inversions

For a construct S of $\mathbf{H}$, $a, b \in H$, write

- $b <_S a$ iff the unique path from the node containing b to the root of S passes through the node containing a, and
- $a =_S b$ iff the same node of S contains a and b,

and define

$$\mathrm{Inv}_{\mathbf{H}}(S) \text{ (resp. } \widetilde{\mathrm{Inv}}_{\mathbf{H}}(S)) := \{(a, b) \in H \times H \mid a < b \text{ and } b <_S a \text{ (resp. } a =_S b)\}.$$

Definition

$\mathbf{H}$ is right-filled if, $\forall a, b, c \in H$ s.t. $a < b < c$ and $\forall X \in \mathbf{H}$ s.t. $a, c \in X$, there exists a hyperedge $Y \in \mathbf{H}$ such that $\{b, c\} \subseteq Y \subseteq (X \backslash \{a\}) \cup \{b\}$.

Theorem (Curien–DO–Obradovic, 26+)

If $\mathbf{H}$ is a right-filled hypergraph, then, for any two constructs S, T of $\mathbf{H}$, it holds that $S \leqslant T$ if and only if $\mathrm{Inv}_{\mathbf{H}}(S) \subseteq \mathrm{Inv}_{\mathbf{H}}(T)$ and $\widetilde{\mathrm{Inv}}_{\mathbf{H}}(S) \subseteq \widetilde{\mathrm{Inv}}_{\mathbf{H}}(T) \cup \mathrm{Inv}_{\mathbf{H}}(T)$.

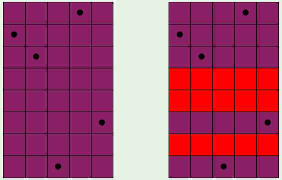
Link with the shuffle products

Outline

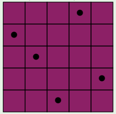
Vertical shuffle

$$\sigma \sqcup\!\sqcup_v \tau = \sum_{\substack{\mathrm{std}(s)=\sigma \\ \mathrm{std}(t)=\tau}} s.t,$$

Example:

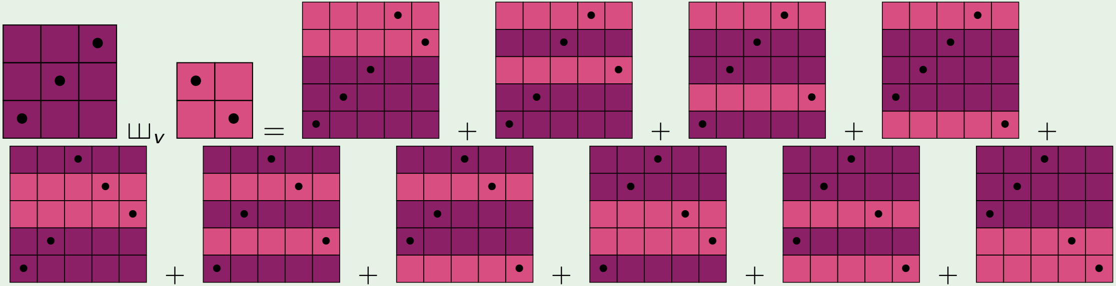


std(76183) = 43152



Example:

$123 \sqcup\!\sqcup_v 21 = 12354 + 12453 + 13452 + 23451 + 12543 + 13542 + 23541 + 14532 + 24531 + 34521$



Link between the shuffle of permutations and weak Bruhat order

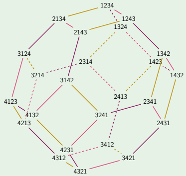
Proposition (Loday-Ronco 02)

There exists two (duplicial) products on permutations $\lhd$ and $\rhd$ s.t. for any permutations $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$:

$$\sigma \sqcup\!\sqcup_v \tau = \sum_{\sigma \rhd \tau \leqslant p \leqslant \sigma \lhd \tau} p,$$

where $\leqslant$ is the weak Bruhat order and $\sigma \lhd \tau = \sigma.(\tau + p)$ and $\sigma \rhd \tau = (\sigma + q).\tau$.

Example:



$$21 \sqcup\!\sqcup 12 = \sum_{4312 \leqslant p \leqslant 2134} p$$

$$= 4312 + 4213 + 4123 + 3214 + 3124 + 2134$$

Shuffles of packed words/surjections [Perm : Chapoton 2000, WQSym: Hivert-Novelli-Thibon ≡2000, NQSym* : Bergeron-Zabrocki, 2005]

Definition

A packed word is a word w on $\mathbb{N}$ such that the set of letters in w is an interval $[\![1;k]\!]$. Equivalently, it is a surjection.

$$u \sqcup\!\sqcup v = \sum_{\substack{\mathrm{pack}(\alpha)=u \\ \mathrm{pack}(\beta)=v}} \alpha\beta,$$

Example :

$$11 \sqcup\!\sqcup 221 = 22221 + 33221 + 22331 + 11221 + 11332$$

Proposition (Palacios-Ronco 02)

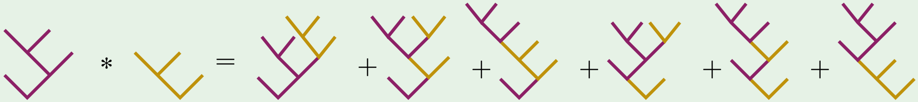
The shuffle product of packed words is the sum of element in an interval of this order.

Shuffles of planar binary trees [Loday-Ronco 1998]

Denoted by $T = \underset{\vee}{T_l \quad T_r}$ a planar binary tree, Loday-Ronco introduced the following product in 1998:

$$T * S = \underset{\vee}{T_l \quad T_r * S} + \underset{\vee}{T * S_l \quad S_r} .$$

Example

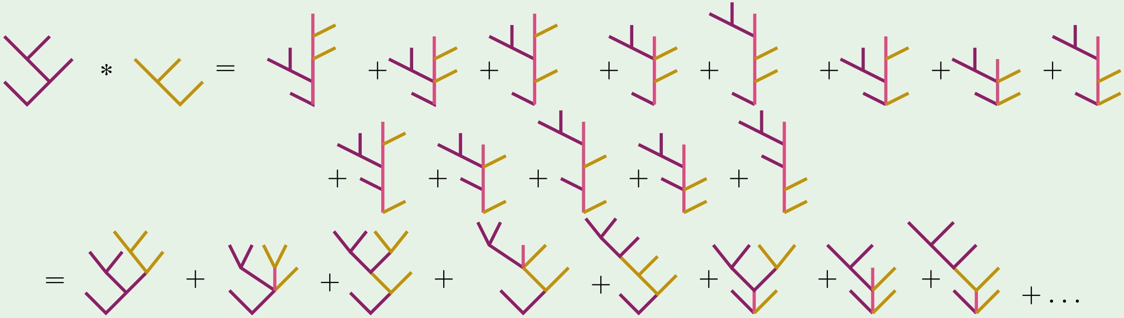


Proposition (Loday-Ronco 2002)

The shuffle product of two planar binary trees S and T can be expressed as a sum of elements in an interval whose bounds are given by some (duplicial) operations on S and T.

Shuffles of planar trees [Loday-Ronco, 2004; Chapoton, 2002]

Example



Proposition (Palacios-Ronco 02)

The shuffle product of planar trees is the sum of element in an interval of this order.

Heuristics for a shuffle product

For $S = A(S_1, \ldots, S_m)$ and $T = B(T_1, \ldots, T_n)$ two constructs of $\mathbf{H}^{\mathcal{X}}$ and $\mathbf{H}^{\mathcal{Y}}$ respectively

$S * T$ as a linear combination of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** A**,** or **root** B**,** or **root** $A \cup B$**.**

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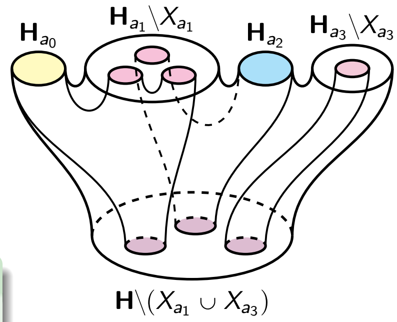
First problem in hypercubes

$$1 * (2 * 3) = 1 * \left(\overset{3}{\textcircled{2}} + \overset{2}{\textcircled{3}} + \textcircled{2\ 3} \right) = 3 \times \overset{3\ 2}{\textcircled{1}} + \dots \neq (1 * 2) * 3$$

Strict teams [Curien–D.O.–Obradovic, 25]

A preteam is a strict team
if the connected components obtained by deleting a subset X_a to every hypergraph $\mathbf{H}_a$ are

- in $\mathfrak{U}$
- and included in the connected components of $\mathbf{H} \setminus \left(\bigcup_{a \in A} X_a\right)$.



$(X_{a_0} = X_{a_2} = \varnothing)$

Examples:

Associahedra, Permutohedra, Restrictohedra, ...

Shuffle product

$$*(\delta) = \sum_{\varnothing \subset B \subseteq A} q^{|B|-1} \Big(\bigcup_{b \in B} X_b\Big)\big(*(\delta_1^B), \ldots, *(\delta_{n_B}^B)\big),$$

Proposition (Ronco 12 ; Curien–D.O.–Obradovic 26+)

For any strict ordered associative clan, the shuffle product is the sum of elements in an interval in the generalised flip order.

Thank you very much for your attention !