From pre-Lie to post-Lie operads through hypertrees

Bérénice Delcroix-Oger (joint work with Clément Dupont (IMAG))

IMAG
INSTITUT MONTPELLIERAIN
ALEXANDER GROTHENDIECK

UNIVERSITÉ MONTPELLIER

ALPE
Thursday, March 30th 2023
https://oger.perso.math.cnrs.fr/expose/ALPE_Oger.pdf

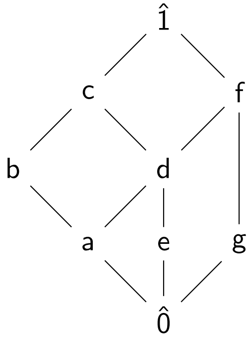
Outline

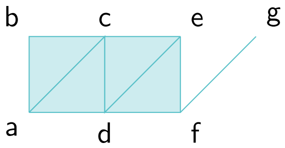
Prehistory : Partition posets

Outline

Poset cohomology

To any bounded poset P can be associated its order complex (nerve), a simplicial set whose simplices are the k-chains $a_0 < \ldots < a_k$ in $P \backslash \{\hat{0}, \hat{1}\}$. The (co)homology of P is the cohomology of its order complex.





What about unbounded posets ?

→Add bounds to delete them !

What about unbounded posets ?

→Add bounds to delete them !

Another definition for the cohomology

$$c^k = \mathbb{C}.\{a_0 < \ldots < a_k | a_0 \text{ minimal and } a_k \text{ maximal}\}$$

In particular, if P is bounded,

$$h^n(P) \simeq \widetilde{H}^{n-2}(P \backslash \{\hat{0}, \hat{1}\}).$$

Posets of (set) partitions Π_V

Partitions of a set V :

$$\{V_1, \dots, V_k\} \models V \Leftrightarrow V = \bigsqcup_{i=1}^{k} V_i, V_i \cap V_j = \varnothing \text{ for } i \neq j$$

Partial order on set partitions of a set V :

$$\{V_1, \dots, V_k\} \leqslant \{V'_1, \dots, V'_p\} \Leftrightarrow \forall i \in \{1, p\}, \exists j \in \{1, k\} \text{ s.t. } V'_i \subseteq V_j$$

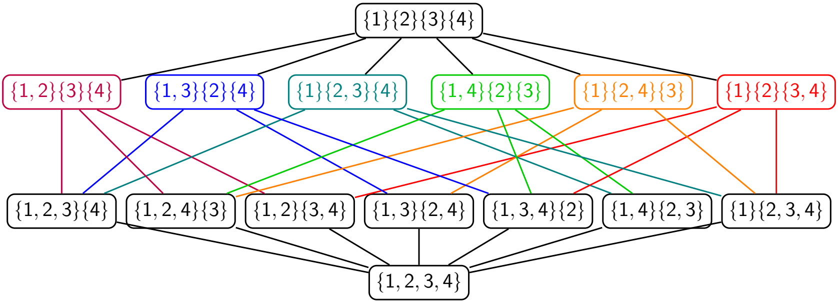
Posets of (set) partitions Π_V

Partitions of a set V :

$$\{V_1, \ldots, V_k\} \models V \Leftrightarrow V = \bigsqcup_{i=1}^{k} V_i, V_i \cap V_j = \emptyset \text{ for } i \neq j$$

Partial order on set partitions of a set V :

$$\{V_1, \ldots, V_k\} \leqslant \{V'_1, \ldots, V'_p\} \Leftrightarrow \forall i \in \{1, p\}, \exists j \in \{1, k\} \text{ s.t. } V'_i \subseteq V_j$$



Operations on posets

If P and Q are two posets, their cartesian product is the set $P \times Q$ endowed with the following partial order :

$$(p, q) \leqslant_{P\times Q} (p', q') \Leftrightarrow p \leqslant_P p', q \leqslant_Q q'$$

Operations on posets

If P and Q are two posets, their cartesian product is the set $P \times Q$ endowed with the following partial order :

$$(p, q) \leqslant_{P \times Q} (p', q') \Leftrightarrow p \leqslant_P p', q \leqslant_Q q'$$

Two posets P and Q are isomorphic if there exists an order-preserving bijection $f : P \to Q$, i.e. a bijection f such that $f(a) \leqslant_Q f(b)$ iff $a \leqslant_P b$.

Operations on posets

If P and Q are two posets, their cartesian product is the set $P \times Q$ endowed with the following partial order :

$$(p, q) \leqslant_{P \times Q} (p', q') \Leftrightarrow p \leqslant_P p', q \leqslant_Q q'$$

Two posets P and Q are isomorphic if there exists an order-preserving bijection $f : P \to Q$, i.e. a bijection f such that $f(a) \leqslant_Q f(b)$ iff $a \leqslant_P b$.
Let $\pi \in \Pi_n$, $\pi = \{V_1, \ldots, V_k\}$

Lemma

The following isomorphisms hold :

$$[\pi, 1_{\Pi_n}] \simeq \prod_{i=1}^{k} \Pi_{|V_k|} \qquad [0_{\Pi_n}, \pi] \simeq \Pi_k$$

Cohomology

Proposition (Hanlon, 81 ; Stanley, 82 ; Joyal 85)

The partition poset Π_n has a unique (co)homology group whose dimension is given by :

$$\mu(\Pi_n) = (n-1)!$$

Moreover, the action of the symmetric group on this homology group is the same as the action on $\mathrm{Lie(n)} \otimes_{\mathfrak{S}_n} \mathrm{sgn}$, *where* sgn *is the signature representation.*

$\mathrm{Lie}(2) = \mathbb{C}.\{[1;2]\}$ with $[1;2] = -[2;1]$

$\mathrm{Lie}(3) = \mathbb{C}.\{[[1;2];3],[[1;3];2]\}$ with
$[[1;2];3] + [[2;3];1] + [[3;1];2] = 0$ (Jacobi relation)

$\mathrm{Lie}(n) = \mathbb{C}.\{[\ldots[1;\sigma(2)]\sigma(3)]\ldots\sigma(n)], \sigma \in \mathfrak{S}(\{2,\ldots,n\})\}$ [Reutenauer]

Levelled (co)bar construction [Fresse, 02]

$$\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\}$$

$$\{1,5\}\{2\}\{3\}\{4\}\{6\}\{7\}\{8\}\{9\}$$

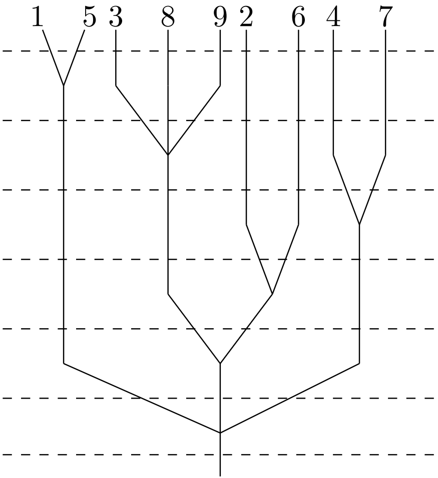
$$\{1,5\}\{2\}\{3,8,9\}\{4\}\{6\}\{7\}$$

$$\{1,5\}\{2\}\{3,8,9\}\{4,7\}\{6\}$$

$$\{1,5\}\{2,6\}\{3,8,9\}\{4,7\}$$

$$\{1,5\}\{2,3,6,8,9\}\{4,7\}$$

$$\{1,2,3,4,5,6,7,8,9\}$$



Levelled (co)bar construction [Fresse, 02]

$$\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\}$$

$$\{1,5\}\{2\}\{3\}\{4\}\{6\}\{7\}\{8\}\{9\}$$

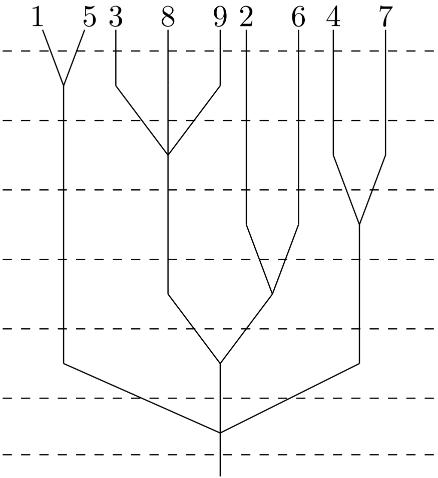
$$\{1,5\}\{2\}\{3,8,9\}\{4\}\{6\}\{7\}$$

$$\{1,5\}\{2\}\{3,8,9\}\{4,7\}\{6\}$$

$$\{1,5\}\{2,6\}\{3,8,9\}\{4,7\}$$

$$\{1,5\}\{2,3,6,8,9\}\{4,7\}$$

$$\{1,2,3,4,5,6,7,8,9\}$$



To keep in mind

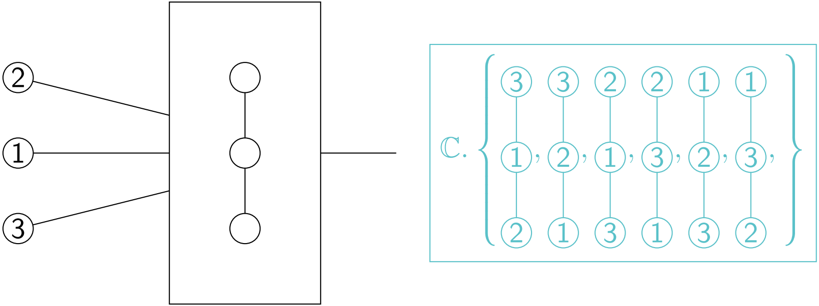
There is an algebraic explanation of why the operad appear in cohomology, linked with cobar construction and Koszul duality

What are species ?

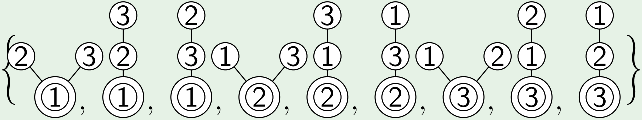
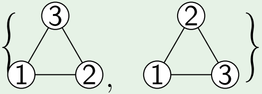
Definition (Joyal, 80s)

A species F is a functor from Bij to Vect. To a finite set S, the species F associates a vector space $F(S)$ independent from the nature of S.

Species = Construction plan, such that the vector space obtained is invariant by relabeling

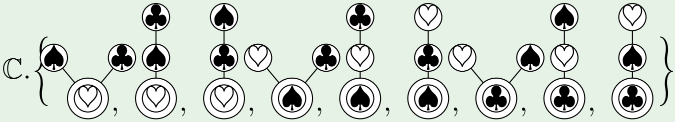
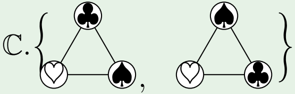


Examples of species

- $\mathbb{C}.\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (species of lists $\mathbb{L}$ on $\{1,2,3\}$)
- $\mathbb{C}.\,\{\{1,2,3\}\}$ (species of non-empty sets $\mathbb{E}^{+}$)
- $\mathbb{C}.\,\{\{1\},\{2\},\{3\}\}$ (species of pointed sets $\mathbb{E}^{\bullet}$)
- $\mathbb{C}.$  (species of Cayley trees $\mathbb{T}$)
- $\mathbb{C}.$  (species of cycles)

These sets are the image by species of the set $\{1,2,3\} =: [\![3]\!]$.

Examples of species

- $\mathbb{C}.\{(\heartsuit,\spadesuit,\clubsuit),(\heartsuit,\clubsuit,\spadesuit),(\spadesuit,\heartsuit,\clubsuit),(\spadesuit,\clubsuit,\heartsuit),(\clubsuit,\heartsuit,\spadesuit),(\clubsuit,\spadesuit,\heartsuit)\}$ (species of lists $\mathbb{L}$ on $\{\clubsuit,\heartsuit,\spadesuit\}$)
- $\mathbb{C}.\ \{\{\heartsuit,\spadesuit,\clubsuit\}\}$ (species of non-empty sets $\mathbb{E}^+$)
- $\mathbb{C}.\ \{\{\heartsuit\},\{\spadesuit\},\{\clubsuit\}\}$ (species of pointed sets $\mathbb{E}^\bullet$)
-  (species of Cayley trees $\mathbb{T}$)
-  (species of cycles)

These sets are the image by species of the set $\{\clubsuit,\heartsuit,\spadesuit\}$.

Substitution of species ("of")

Proposition

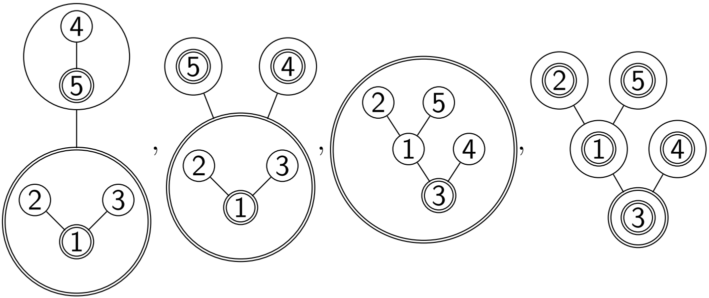
Let F and G be two species. Let us define :

$$(F \circ G)(S) = \bigoplus_{\pi \in \Pi(S)} F(\pi) \otimes \bigotimes_{J \in \pi} G(J),$$

where $\Pi(S)$ runs on the set of partitions of S.

$\mathbb{T} \circ \mathbb{T}([\![5]\!]) =$

$\mathbb{C}.\Bigg\{$



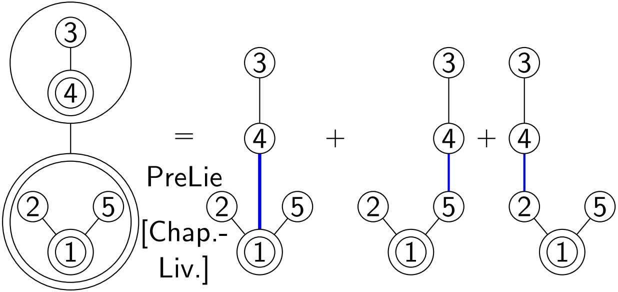
$, \dots \Bigg\}$

Operads

A (symmetric) operad $\mathcal{O}$ is

- a species $\mathcal{O}$ with an associative composition

$$\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$$



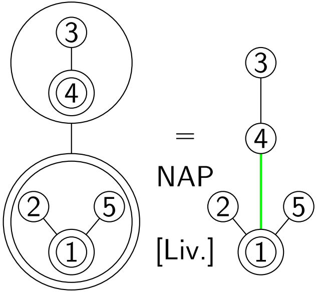
- and a unit $i : I \to \mathcal{O}$, where I is the singleton species $(I(S) = \delta_{|S|=1}\mathbb{C})$.
- To each kind of algebra is associated an operad.

Operads

A (symmetric) operad $\mathcal{O}$ is

- a species $\mathcal{O}$ with an associative composition

$$\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$$



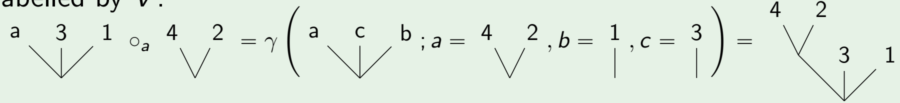
- and a unit $i : I \to \mathcal{O}$, where I is the singleton species ($I(S) = \delta_{|S|=1}\mathbb{C}$).
- To each kind of algebra is associated an operad.

Free operad

Let M be $\mathfrak{S}$-module. The free operad over M is the operad whose underlying species associate to any finite set V the set of rooted trees whose leaves are labelled by V and whose inner vertices are labelled by an element of M, with substitution given by grafting on leaves.

Mag operad

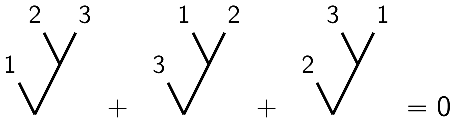
When $M = \mathbb{C}.\{(1,2),(2,1)\}$, the free operad is called Magmatic operad. The species $\mathcal{M}ag(V)$ is the species of planar binary trees with leaves labelled by V.



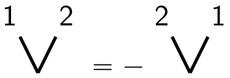
Any operad can be described as a quotient of a free operad.

Lie operad

Lie operad encodes Lie algebra. Its underlying vector space is obtained as a quotient of the Magmatic operad's vector spaces with the Jacobi relations



and the anti-symmetry



Proposition

The vector space of n-ary operations of Lie operad has dimension $\mathrm{Lie}(n) = (n-1)!$ *(comb).*

Ancient history : Hypertree posets

Outline

1. Prehistory : Partition posets
2. Ancient history : Hypertree posets
3. Modern era : Post-Lie and pre-Lie operads and the hypertree posets

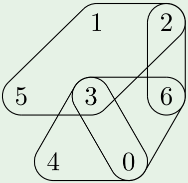
Hypergraphs

Definition (Berge)

A hypergraph (on a set V) is an ordered pair (V, E) where :

- V is a finite set (vertices)
- E is a collection of subsets of cardinality at least two of elements of V (edges).

Example of a hypergraph on $[1; 7]$



Walk on a hypergraph

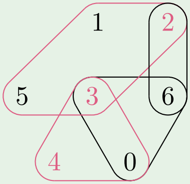
Definition

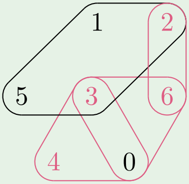
Let $H = (V, E)$ be a hypergraph.
A walk from a vertex or an edge d to a vertex or an edge f in H is an alternating sequence of vertices and edges beginning by d and ending by f :

$$(d, \ldots, e_i, v_i, e_{i+1}, \ldots, f)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.
The length of a walk is the number of edges and vertices in the walk.

Examples of walks





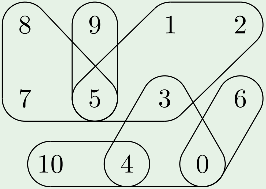
Hypertrees

Definition

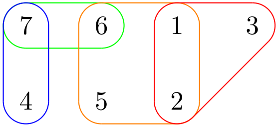
A hypertree is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

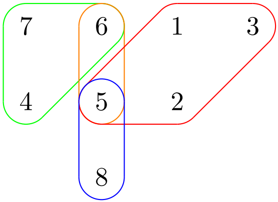
- there exists a walk from v to w in H with distinct edges e_i, (H is connected),
- and this walk is unique, (H has no cycles).

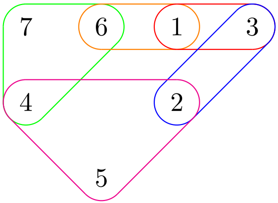
Example of a hypertree

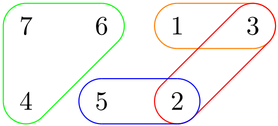


Examples and counter-examples : Which one(s) is/are a hypertree(s) ?









The hypertree poset

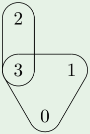
Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

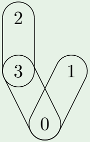
$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

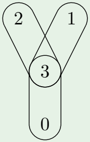
Example with hypertrees on four vertices



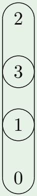
$\preceq$



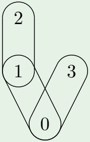
,

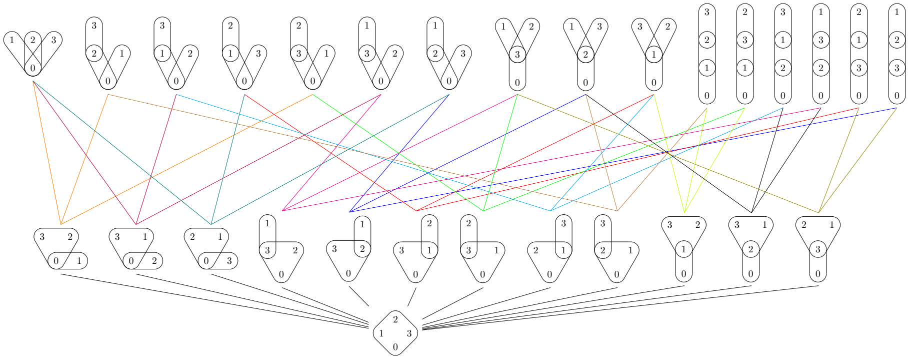


,



but not





- $\widehat{\mathrm{HT}_n}$ = augmented hypertree poset on $[\mathbf{0}, n]$.
- HT_n = hypertree poset on $[\mathbf{0}, n]$.
- For a a tree in HT_n, HT_n^a = maximal interval in hypertree poset on $[\mathbf{0}, n]$ between $\hat{0}$ and a.

Intervals in the hypertree posets

Proposition (McCammond–Meier, 2004)

Let H be a hypertree on n vertices and a be a tree such that $H \leqslant a$. The following isomorphisms hold :

$$[0_{\mathrm{HT}_n}, H] \simeq \prod_{v \in V(H)} \Pi_{\deg(v)} \qquad [H, a] \simeq \prod_{e \in E(H)} \mathrm{HT}_e^{a_{|e}}.$$

In particular, $\mathrm{HT}_n^a = \prod_{v \in V(a)} \Pi_{\deg(v)}$.
Moreover, the order complex of $\widehat{\mathrm{HT}}_n$ *is homotopic to a wedge of spheres of same dimension (Cohen-Macaulay).*

Möbius number of the hypertree posets

Proposition (McCammond-Meier, 2004)

The Möbius number of $\widehat{\mathrm{HT}}_n$ is given by :

$$\mu(\widehat{\mathrm{HT}}_n) = (-1)^{n-1} n^{n-1}$$

Proposition (Conjecture of Chapoton, BO 2013)

The action of the symmetric group on the unique homology group of $\widehat{\mathrm{HT}}_n$ is the same as the action of the symmetric group on Cayley trees.

Möbius number of the hypertree posets

Proposition

The Möbius number of HT_n *is given by :*

$$\mu(\mathrm{HT}_n) = (-1)^n \frac{(2n-1)!}{n!}$$

A006963 Number of planar embedded labeled trees with n nodes: (2n-3)!/(n-1)! for n >= 2, a(1) = 1. 28

(Formerly M3076)

1, 1, 3, 20, 210, 3024, 55440, 1235520, 32432400, 980179200, 33522128640, 1279935820800, 53970627110400, 2490952020480000, 124903451312640000, 6761440164390912000, 393008709555221760000, 24412776311194951680000, 1613955767240110694400000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS For n>1: central terms of the triangle in A173333; cf. A001761, A001813. - Reinhard Zumkeller, Feb 19 2010
Can be obtained from the Vandermonde permanent of the first n positive integers; see A093883. - Clark Kimberling, Jan 02 2012
All trees can be embedded in the plane, but "planar embedded" means that orientation matters but rotation doesn't. For example, the n-star with n-1 edges has n! ways to label it, but rotation removes a factor of n-1. Another example, the n-path has n! ways to label it, but rotation removes a factor of 2. - Michael Somos, Aug 19 2014

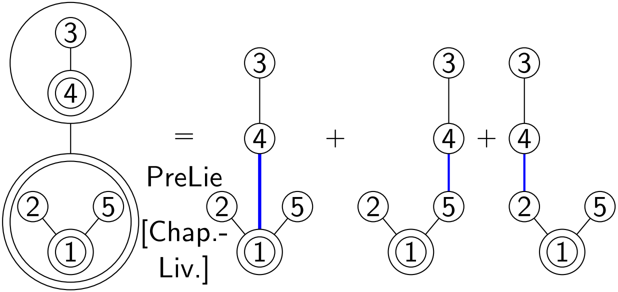
REFERENCES N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

Modern era : Post-Lie and pre-Lie operads and the hypertree posets

Outline

Pre-Lie operad [Chapoton–Livernet, 00 ; Dzhumadil'daev–Löfwall, 02]

PreLie(V) is spanned by Cayley trees with nodes labeled by V. The substitution of a tree t inside a node v is given by the sum over all the ways to graft each child of v on a node of t.



Proposition

The vector space of n-ary operations of Pre-Lie operad has dimension Pre-Lie$(n) = n^{n-1}$.

Pre-Lie algebras

- The pre-Lie product $\leftharpoondown$ satisfies the following relation for any elements x, y and z :

$$(x \leftharpoondown y) \leftharpoondown z - x \leftharpoondown (y \leftharpoondown z) = (x \leftharpoondown z) \leftharpoondown y - x \leftharpoondown (z \leftharpoondown y).$$

Pre-Lie algebras

- The pre-Lie product $\leftharpoondown$ satisfies the following relation for any elements x, y and z :

$$(x \leftharpoondown y) \leftharpoondown z - x \leftharpoondown (y \leftharpoondown z) = (x \leftharpoondown z) \leftharpoondown y - x \leftharpoondown (z \leftharpoondown y).$$

The question you all dying to ask

Why is it called pre-Lie ?

Pre-Lie algebras ... and post-Lie algebras !

- The pre-Lie product $\leftharpoondown$ satisfies the following relation for any elements x, y and z :

$$(x \leftharpoondown y) \leftharpoondown z - x \leftharpoondown (y \leftharpoondown z) = (x \leftharpoondown z) \leftharpoondown y - x \leftharpoondown (z \leftharpoondown y).$$

- The post-Lie products $\lhd$ and $\{\ ;\ \}$ satisfy

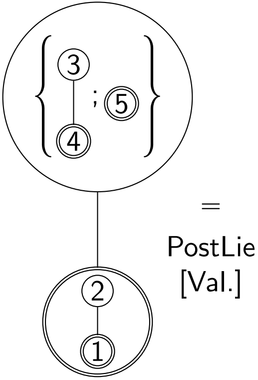
$\{\ ;\ \}$ is a Lie bracket

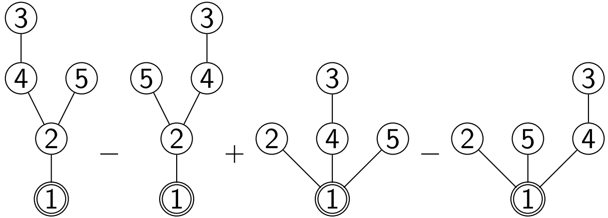
$$(x \lhd y) \lhd z - x \lhd (y \lhd z) - (x \lhd z) \lhd y + x \lhd (z \lhd y) = x \lhd \{y, z\}$$

$$\{x, y\} \lhd z = \{x \lhd z, y\} + \{x, y \lhd z\}$$

Post-Lie operad [Vallette, 07 ; Munthe-Kaas–Wright, 08]

The underlying vector space PostLie(V) of post-Lie operad is spanned by Lie brackets of planar trees with nodes labeled by V. The substitution of a tree t inside a node v is given by the sum over all the way to graft each child of v to the right of a node of t (planar pre-Lie product).





Proposition

The vector space of n-ary operations of Post-Lie operad has dimension $\text{Post-Lie}(n) = \frac{(2n-1)!}{n!}$.

Motivation : Levelled (co)bar construction [Fresse, 02]

$\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\}$

$\{1,5\}\{2\}\{3\}\{4\}\{6\}\{7\}\{8\}\{9\}$

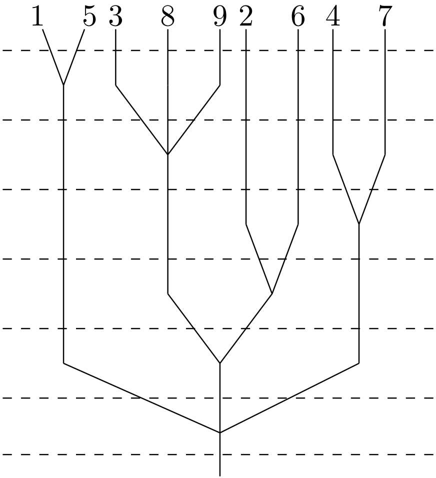
$\{1,5\}\{2\}\{3,8,9\}\{4\}\{6\}\{7\}$

$\{1,5\}\{2\}\{3,8,9\}\{4,7\}\{6\}$

$\{1,5\}\{2,6\}\{3,8,9\}\{4,7\}$

$\{1,5\}\{2,3,6,8,9\}\{4,7\}$

$\{1,2,3,4,5,6,7,8,9\}$



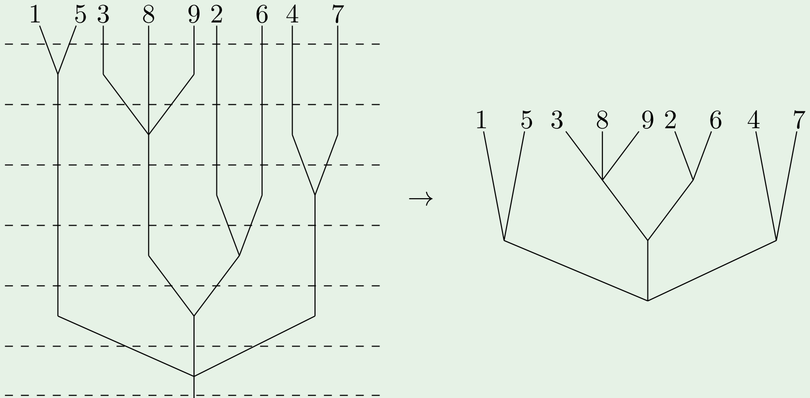
Nested sets [De Concini–Procesi, 95]

Problem

There are no operadic structure on the leveled cobar construction, but there is one on the cobar construction !

Solution :

Forget about the levels !



Merge trees

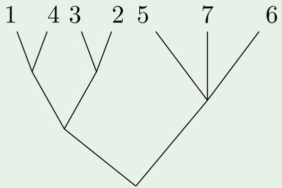
Definition

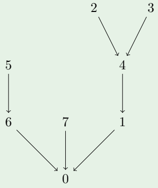
For a nonnegative integer n, a n-merge tree is

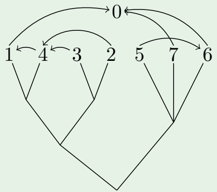
- a (non planar) reduced (i.e. with no inner vertex of arity 1) rooted tree B with leaves labelled by a set $\{1, \dots, n\}$, called bottom tree
- a (non planar) rooted tree T with vertices labelled by $\{0, 1, \dots, n\}$, rooted in 0, called top tree

such that for any vertex v in B, the set of outgoing edges coming from a leaf above v in B forms a subtree of T.

Example

$B =$ 

$T =$ 

$MT =$  .

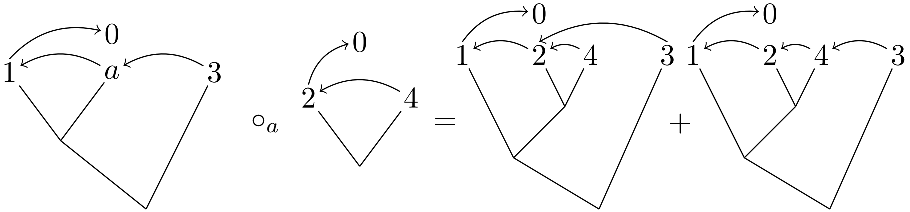
Merge trees

Proposition

The complex of merge trees is homotopy equivalent to the order complex of the poset HT_n.

The operadic composition of a merge tree b in a node v is as follows :

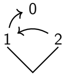
- the children of v in T are grafted to some nodes in b (pre-Lie type magmatic composition)
- the bottom tree of b is grafted at the place of the leaf v (usual magmatic composition)

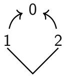


Operadic structure on the cohomology of the nested set complex (aka. post-Lie !)

Let us consider the map

$$\text{Post-Lie} \xrightarrow{\phi} H^{\bullet}(HT_{\bullet})$$

$1 \lhd 2 \mapsto$ 

$\{1; 2\} \mapsto$ 

Theorem (DO–Dupont, 23+)

The map ϕ is an operad isomorphism. The operadic structure on the cohomology of the hypertree poset is then isomorphic to the suspension of post-Lie operad.

Left-post-Lie module on Cayley trees

$$1 \lhd T = 1 \leftharpoondown T,$$
$$(G \leftharpoondown D) \lhd T = (G \lhd T) \leftharpoondown D + G \leftharpoondown (D \lhd T)$$
$$\{S, T\} = T \leftharpoondown S - S \leftharpoondown T,$$

where $\leftharpoondown$ is the usual pre-Lie product.

Proposition (D.O.–Dupont, 23+)

This defines a left-post-Lie module structure on Cayley trees. This left-post-Lie module is moreover monogenous.

Left-post-Lie module on hyper-merge trees

Definition

For a nonnegative integer n, a hyper-merge-tree is

- a partition π of $\{1, \ldots, n\}$,
- a (non planar) reduced (i.e. with no inner vertex of arity 1) rooted tree B with leaves labelled by π, called bottom tree
- a (non planar) hypertree T with edge partition $\{0\} \cup \pi$, rooted in 0, called top tree

such that for any vertex v in B, the set of outgoing edges coming from a leaf above v in B forms a subtree of T.

Proposition (D.O.–Dupont, 23+)

These two left-post-Lie modules are isomorphic.

Still some work left !

- The multi-pointed hypertree poset project to both the hypertree poset and the poset of partitions decorated by Operad ComTrias [Vallette, 07].
- This fits in a more general setting, which applies to examples of families of poset (F_n) endowed with a "good" projection $a : F_n \to \Pi_n$.

Work in progress

In the general setting, describe the operadic structure on the nested set complex associated with the minimal building set itself (and not only on the cohomology).

Thank you for your attention !