# Counting decorated hypertrees

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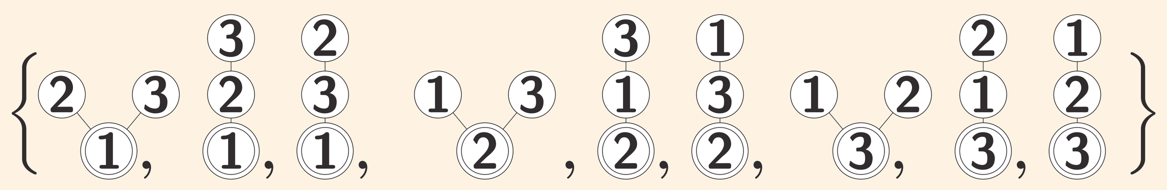
## Species

A **species** $\mathbf{F}$ is a functor from the category of finite sets and bijections to the category of finite sets.

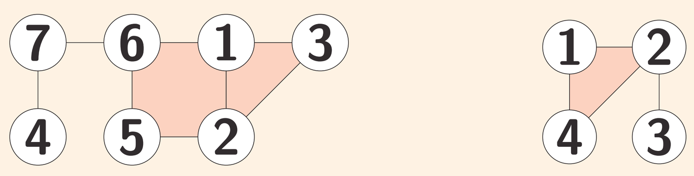
$\{1,2,3\} \mapsto$ 

Example : The species of rooted trees, called the **PreLie** species.

Let $\mathbf{F}$ and $\mathbf{G}$ be two species. One can define the sum, the composition and the product of $\mathbf{F}$ and $\mathbf{G}$. The derivative of $\mathbf{F}$ is given by : $\mathbf{F'(I) = F(I \sqcup \{\bullet\})}$.

## Hypergraphs and hypertrees

A **hypergraph (on a set V)** is an ordered pair $\mathbf{(V, E)}$ where $\mathbf{V}$ is a finite set and $\mathbf{E}$ is a collection of parts of $\mathbf{V}$ of cardinality at least two. The elements of $\mathbf{V}$ are called **vertices** and those of $\mathbf{E}$ are called **edges**.



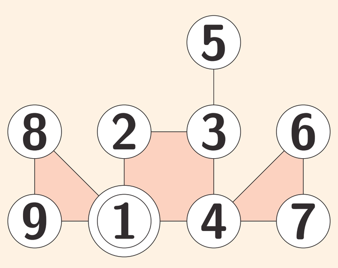
There are two walks between **5** and **3** in the left-side hypergraph : it is not an hypertree

A **hypertree** is a non empty hypergraph $\mathbf{H}$ such that, given any vertices $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{H}$, there exists one and only one walk from $\mathbf{v}$ to $\mathbf{w}$ in $\mathbf{H}$ with distinct edges $\mathbf{e_i}$, i.e. $\mathbf{H}$ is **connected** and has **no cycles**.

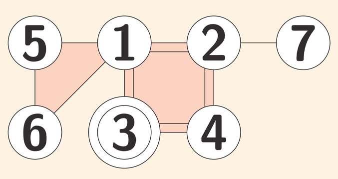
## Motivation

To study groups of automorphisms of free groups $\mathbf{F_n}$, McCammond et Meier used a weight $(|\mathbf{e}| - \mathbf{1})^{|\mathbf{e}|-2}$ on edges $\mathbf{e}$ of hypertrees. We give an interpretation of this weight in terms of decorated hypertrees and 2-colored rooted trees.

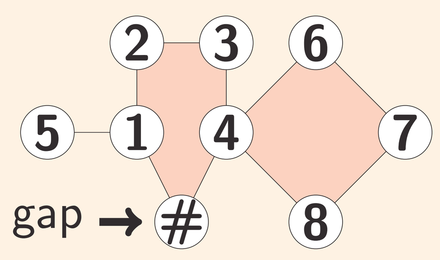
## Pointing and rooting



A rooted hypertree (distinguished vertex)



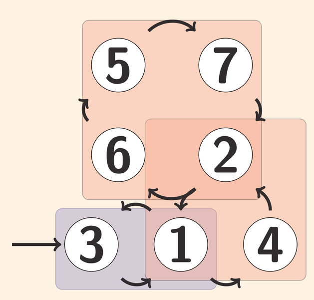
An edge-pointed rooted hypertree (distinguished vertex in a distinguished edge)



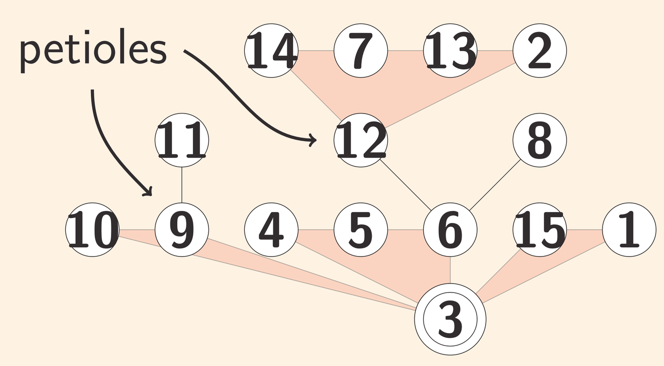
A hollow hypertree (additional vertex in one and only one edge : the gap)

## Decorated hypertrees

- Given a species $\mathcal{S}$, a **decorated (edge-pointed/ rooted/ hollow) hypertree** is obtained from a (edge-pointed/ rooted/ hollow) hypertree $\mathbf{H}$ by choosing for every edge $\mathbf{e}$ of $\mathbf{H}$ an element of $\mathcal{S}(\mathbf{V_e})$, where $\mathbf{V_e}$ is the set of vertices in the edge $\mathbf{e}$.

pointed edge → 

Decorated edge-pointed hypertree by the species of cycles.



Decorated rooted hypertree by the species of cycles, whose derivative is the species of lists

- Given an edge $\mathbf{e}$ of a rooted (resp. hollow) hypertree $\mathbf{H}$, there is one vertex of $\mathbf{e}$ which is the nearest from the root (resp. the gap) of $\mathbf{H}$ in $\mathbf{e}$: the **petiole** $\mathbf{p_e}$ of $\mathbf{e}$. Then, a **decorated rooted** (resp. **hollow**) **hypertree** is obtained from $\mathbf{H}$ by choosing for every edge $\mathbf{e}$ of $\mathbf{H}$ an element in $\mathcal{S}'(\mathbf{V_e} - \{\mathbf{p_e}\})$.
- When $\mathcal{S} = \widehat{\mathbf{PreLie}}$, the edges of the decorated hypertrees contain a vertex (or a gap) and a rooted tree because $\mathcal{S}' = \mathbf{PreLie}$.

## Relations between hypertrees species

The species $\mathcal{H}_{\widehat{\mathrm{PreLie}}}$, $\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}}$, $\mathcal{H}^{a}_{\widehat{\mathrm{PreLie}}}$, $\mathcal{H}^{pa}_{\widehat{\mathrm{PreLie}}}$ and $\mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}}$ satisfy:

$$\mathcal{H}_{\widehat{\mathrm{PreLie}}} + \mathcal{H}^{pa}_{\widehat{\mathrm{PreLie}}} = \mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} + \mathcal{H}^{a}_{\widehat{\mathrm{PreLie}}}. \text{ (Dissymetry principle)}$$

The proof uses the notion of **center** of a hypertree.

$$\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} = \mathbf{X} \times \mathcal{H}'_{\widehat{\mathrm{PreLie}}},$$

$$t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} = \mathbf{X} + \mathbf{X} \times \mathbf{Comm} \circ \left(\mathbf{t} \times \mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}}\right),$$

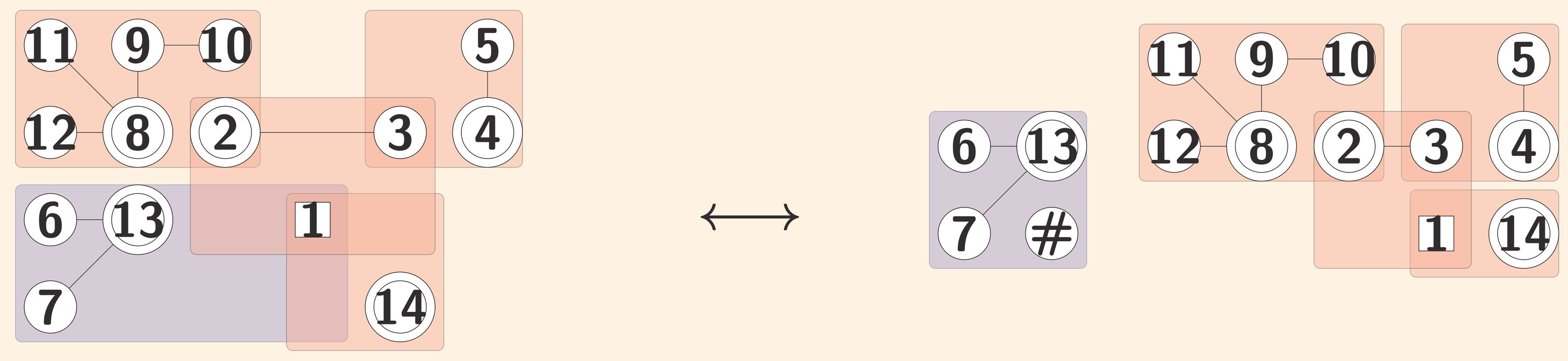
$$\mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}} = \mathbf{PreLie} \circ t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}},$$

$$\mathcal{H}^{a}_{\widehat{\mathrm{PreLie}}} = \widehat{\mathbf{PreLie}} \circ t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}},$$

$$\mathcal{H}^{pa}_{\widehat{\mathrm{PreLie}}} = \mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}} \times t\mathcal{H}^{p}_{\widehat{\mathrm{PreLie}}} = \frac{\mathbf{X}}{\mathbf{t}} \times \left(\mathbf{X}(\mathbf{1} + \mathbf{Comm}) \circ t\mathcal{H}^{c}_{\widehat{\mathrm{PreLie}}}\right).$$

with the weight $\mathbf{W(H = (V, E)) = t^{|E|-1}}$.

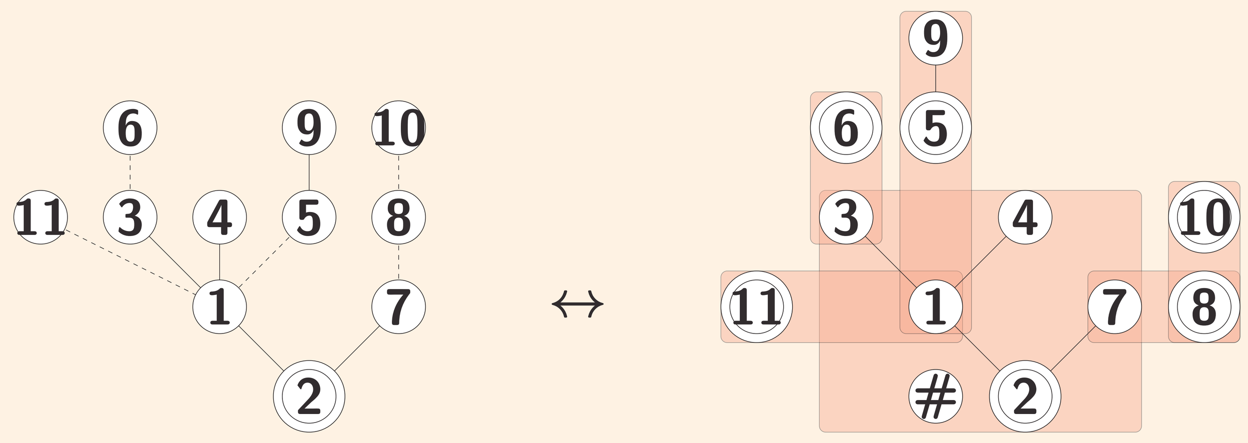
## Example of species isomorphism between hypertrees



Decomposition of the edge-pointed rooted hypertree into a hollow and a rooted hypertrees.

## 2-colored trees

A **2-colored rooted tree** is a rooted tree $\mathbf{(V, E)}$, where $\mathbf{V}$ is the set of vertices and $\mathbf{E \subseteq V \times V}$ is the set of edges decomposed into $\mathbf{E = E_0 \cup E_1}$, with $\mathbf{E_0 \cap E_1} = \emptyset$.



A 2-colored rooted tree (left) and the associated hollow hypertree (right) decorated by $\widehat{\mathbf{PreLie}}$.

## Theorem

- The species of **hollow hypertrees** decorated by $\widehat{\mathbf{PreLie}}$ is isomorphic to the species of 2-coloured rooted trees.
- The species of **rooted hypertrees** decorated by $\widehat{\mathbf{PreLie}}$ is isomorphic to the species of 2-coloured rooted trees such that the edges adjacent to the root are all dashed.
- The species of **rooted edge-pointed hypertrees** decorated by $\widehat{\mathbf{PreLie}}$ is isomorphic to the species of 2-coloured rooted trees such that all the edges adjacent to the root but one are dashed.

## Results on generating series

The generating series of the species of **hollow hypertrees** decorated by $\widehat{\mathbf{PreLie}}$ is given by:

$$\mathbf{S}^{c}_{\widehat{\mathrm{PreLie}}} = \mathbf{x} + \sum_{\mathbf{n \geq 2}} (\mathbf{tn + n})^{\mathbf{n-1}} \frac{\mathbf{x^n}}{\mathbf{n!}}.$$

The generating series of the species of **rooted hypertrees** decorated by $\widehat{\mathbf{PreLie}}$ is given by:

$$\mathbf{S}^{p}_{\widehat{\mathrm{PreLie}}} = \frac{\mathbf{x}}{\mathbf{t}} + \sum_{\mathbf{n \geq 2}} \mathbf{n}(\mathbf{tn + n - 1})^{\mathbf{n-2}} \frac{\mathbf{x^n}}{\mathbf{n!}}.$$

The generating series of the species of **hypertrees** decorated by $\widehat{\mathbf{PreLie}}$ is given by:

$$\mathbf{S}_{\widehat{\mathrm{PreLie}}} = \mathbf{x} + \sum_{\mathbf{n \geq 2}} (\mathbf{tn + n - 1})^{\mathbf{n-2}} \frac{\mathbf{x^n}}{\mathbf{n!}}.$$

The generating series of the species of **rooted edge-pointed hypertrees** decorated by $\widehat{\mathbf{PreLie}}$ is given by:

$$\mathbf{S}^{pa}_{\widehat{\mathrm{PreLie}}} = \mathbf{x} + \sum_{\mathbf{n \geq 2}} \mathbf{n}(\mathbf{n + tn - 1})^{\mathbf{n-3}} (\mathbf{n - 1})(\mathbf{1 + 2t}) \frac{\mathbf{x^n}}{\mathbf{n!}}.$$

The generating series of the species of **edge-pointed hypertrees** decorated by $\widehat{\mathbf{PreLie}}$ is given by:

$$\mathbf{S}^{a}_{\widehat{\mathrm{PreLie}}} = \mathbf{x} + \sum_{\mathbf{n \geq 2}} (\mathbf{n + tn - 1})^{\mathbf{n-3}} (\mathbf{n - 1})(\mathbf{1 + tn}) \frac{\mathbf{x^n}}{\mathbf{n!}}.$$

## Further results

- The generating series $\mathbf{S}^{p}_{\widehat{\mathrm{PreLie}}}$ and $\mathbf{S}_{\widehat{\mathrm{PreLie}}}$ are the same as some series in the article of McCammond and Meier.
- Box trees also help to count other decorated hypertrees.
- There are links with the hypertree poset, using decorations around vertices. (cf. B. Oger, JACO, 2013, DOI:10.1007/s10801-013-0432-2)

## References


- B. Oger, *Decorated hypertrees*, ArXiv : 1209.0941, 2012, (submitted)
- C. Jensen, J. McCammond and J. Meier, *The Euler characteristic of the Whitehead automorphism group of a free product*, Trans. Amer. Math. Soc., 2007, 2577–2595 (electronic).