# Parking trees

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# Outline

# Set partitions, noncrossing partitions and parking functions

# Set partition and noncrossing partitions

## Definition

A (set) partition of $E$ is $\pi = \{\pi_1, \ldots, \pi_k\}$ s.t. :

- $\pi_k \cap \pi_l \neq \varnothing \implies k = l$
- and $\bigcup_{i=1}^{k} \pi_i = E$.

$\Pi_E$ = set of partitions of $E$

Examples :

## Definition (Kreweras, 1972)

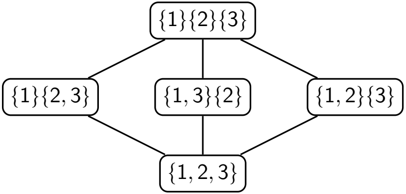
A partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of $\{1, \ldots, n\}$ is non-crossing iff

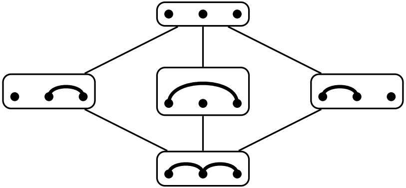
$$\begin{cases} a < b < c < d \\ a, c \in \pi_i \\ b, d \in \pi_j \end{cases} \implies i = j$$

$NC_n$ = set of non-crossing partitions of $\{1, \ldots, n\}$

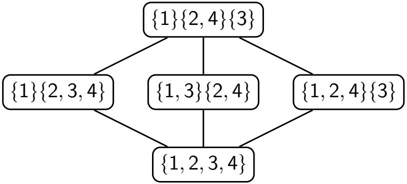
$\rightarrow$ Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$

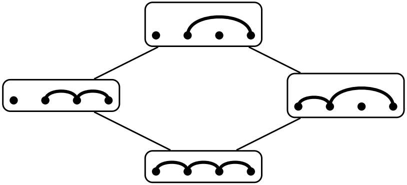
# Partitions and non-crossing partitions poset



# Partitions and non-crossing partitions poset



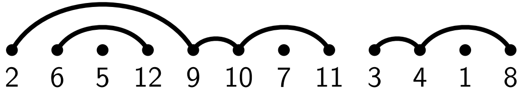



# Non-crossing 2-partitions

**Definition (Edelman, 1980)**

A n.c. 2-partition of size $n$ is a pair $(\pi, \sigma) \in NCP_n \times \mathfrak{S}_n$ s.t.

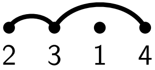
$$\begin{cases} \{b_1, \ldots, b_k\} \in \pi \\ b_1 < b_2 < \ldots < b_k \end{cases} \implies \sigma(b_1) < \sigma(b_2) < \ldots < \sigma(b_k).$$



# 2NCP poset

Covering relation in □ : rearranging labels to respect the increasing condition

Example :




→ $(n+1)^{n-1}$

# Parking function [Konheim-Weiss, 1966]

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |

**Question :**

How to park 6 cars in 6 parking spaces ?

Easy answer : bijection between the parking spaces and cars.

What if you pick at random for each car its place ? Can all cars park ?

→ If yes, parking function !

# Parking function [Konheim-Weiss, 1966]

## Parking function

$f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ s.t. $\bigcup_{j=1}^{i} |f^{-1}(j)| \geqslant i$

## Examples & counter-example : Find the odd one out !

- 1
- 11, 12, 21
- 111, 112, 121, 211, 122, 212, 221, 113, 131, 311, 123, 132, 213, 231, 312, 321
- 1365247
- 4166114
- 153436
- 122333

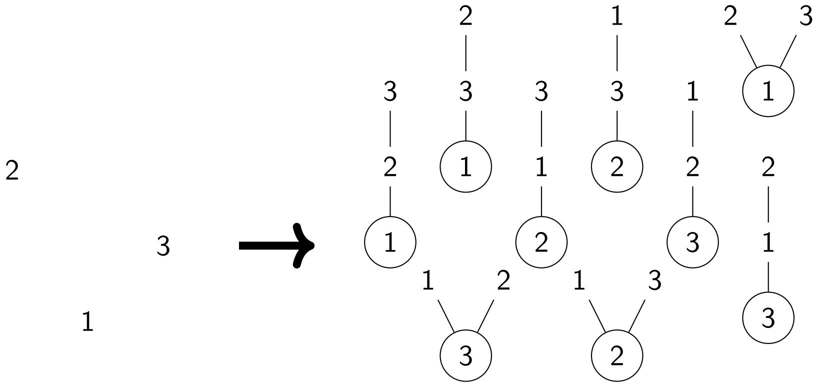
$\to (n+1)^{n-1}$

# Parking trees and species

# Species [Joyal, 1980]

**Definition**

Species F : FinSet → FinSet which associates to a fin. set $I$, the fin. set F($I$), only depending on the cardinality of $I$.



# Species [Joyal, 1980]

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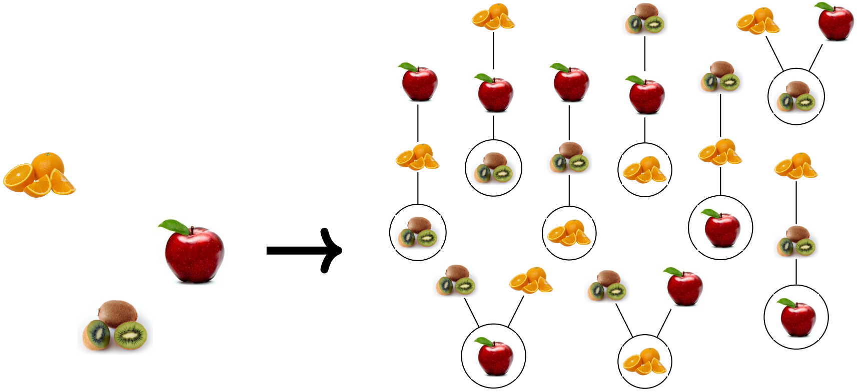
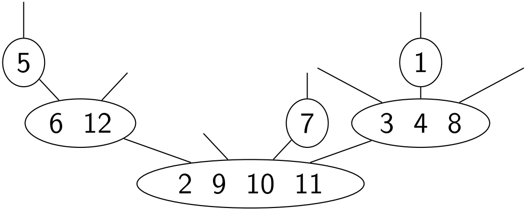


# Parking trees

## Definition

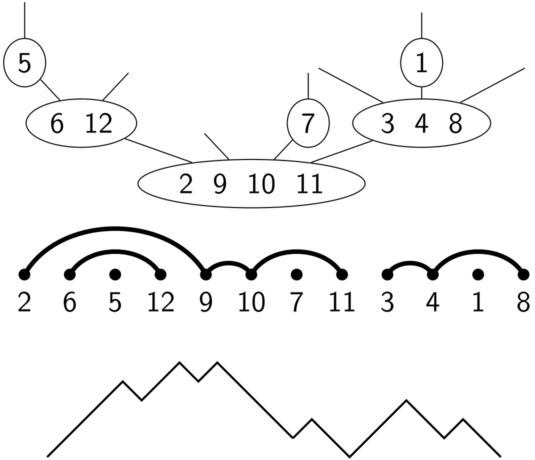
A parking tree on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

- $V \in \Pi_L$,
- $v \in V$ has $|v|$ children.




Why parking ?

# Bijection between 2NCP <-> parking trees



# Why do we need species ?

Let $F$ and $G$ be two species.

- $(F + G)(I) = F(I) \sqcup G(I)$,

- $(F \times G)(I) = \bigsqcup_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$.

### Definition

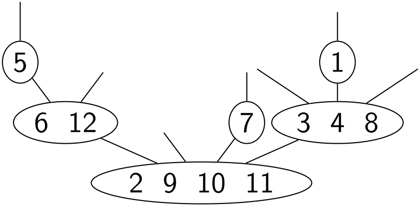
The cycle index series of a species $F$ is the formal power series in an infinite number of variables $\mathfrak{p} = (p_1, p_2, p_3, \ldots)$ defined by:

$$Z_F(\mathfrak{p}) = \sum_{n \geqslant 0} \frac{1}{n!} \left( \sum_{\sigma \in \mathfrak{S}_n} F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \ldots \right),$$

- with $F^\sigma = |\{x \in F(\{1, \ldots, n\}) | \sigma \cdot x = x\}|$
- and $\sigma$ has $\sigma_i$ cycles of length $i$.

Example :

# Back to parking trees

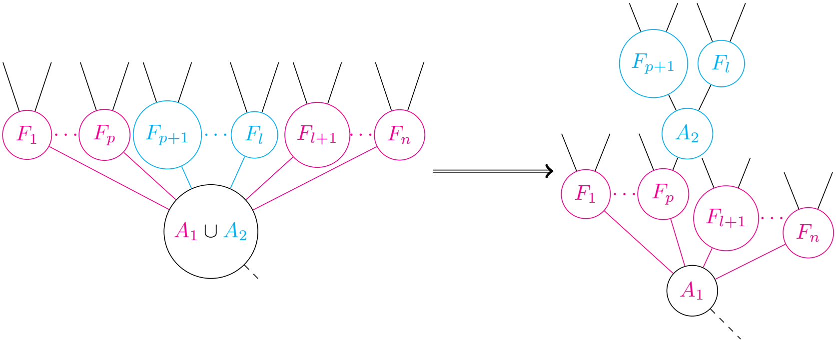


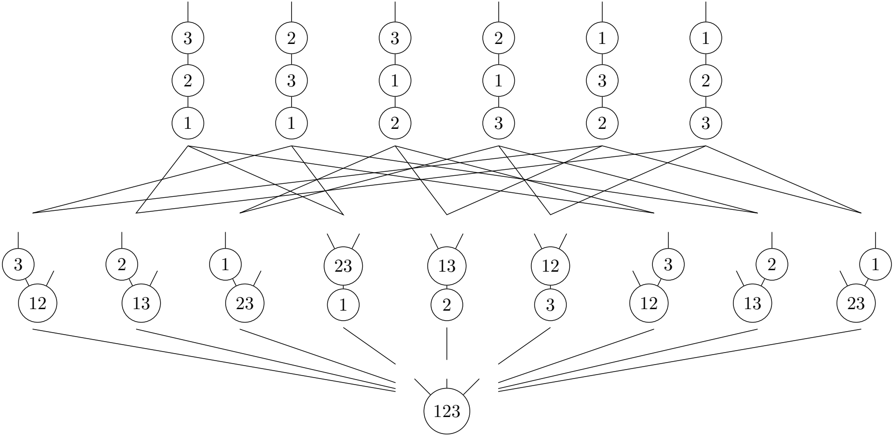

**Proposition (DO, Josuat-Vergès, Randazzo, 20+)**

$$\mathcal{P}_f = \sum_{p \geqslant 1} \mathcal{E}_p \times (1 + \mathcal{P}_f)^p$$

# Parking poset

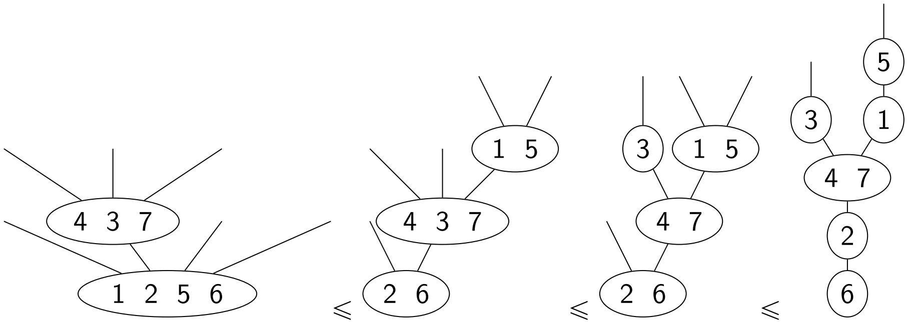
# Order on parking trees



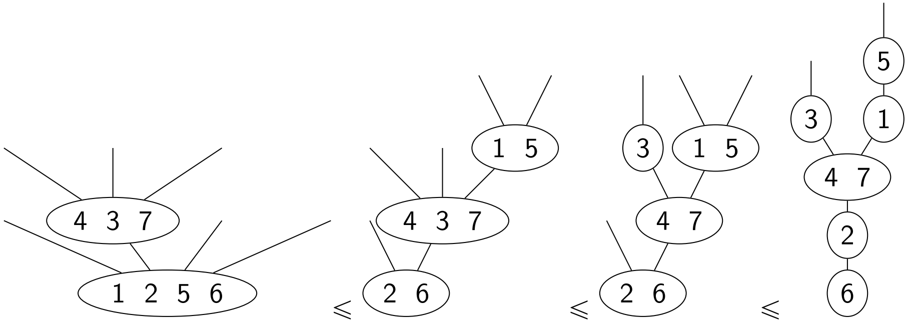

3
2
3
2
1
1
2
3
1
1
3
2
1
1
2
3
2
3
3
2
1
23
13
12
3
2
1
12
13
23
1
2
3
12
13
23
123

# Results

- This poset is a lattice
- When restricting right combs, get the face poset of the permutohedron
- New criterion to prove shellability !
- Enumeration of (weak) k-chains



# $k$-weak chains




**Proposition (DO, Josuat-Vergès, Randazzo, 20+)**

$$\mathcal{C}^{I}_{k,t} = \sum_{p \geqslant 1} \mathcal{C}^{I,p}_{k-1,t} \times \left(t\mathcal{C}^{I}_{k,t} + 1\right)^{p},$$

# $k$-weak chains

**Proposition (DO,Josuat-Vergès,Randazzo, 20+)**

*Chains $\phi_1 \leqslant \cdots \leqslant \phi_k$ in ?$_n$ are in bijection with $k$-parking trees.*
*The number of chains $\phi_1 \leqslant \cdots \leqslant \phi_k$ in ?$_n$ where $\mathrm{rk}(\phi_k) = \ell$ is:*

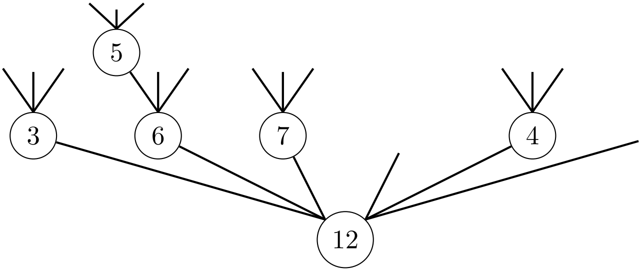
$$\ell! \binom{kn}{\ell} S_2(n, \ell + 1).$$

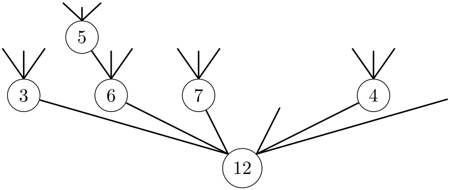
# $k$-parking tree

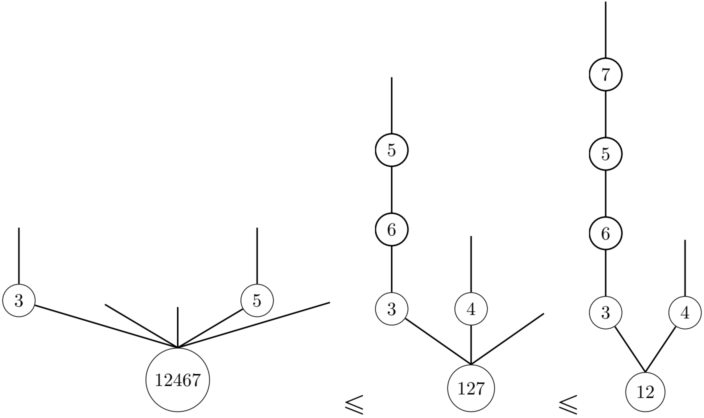
## Definition

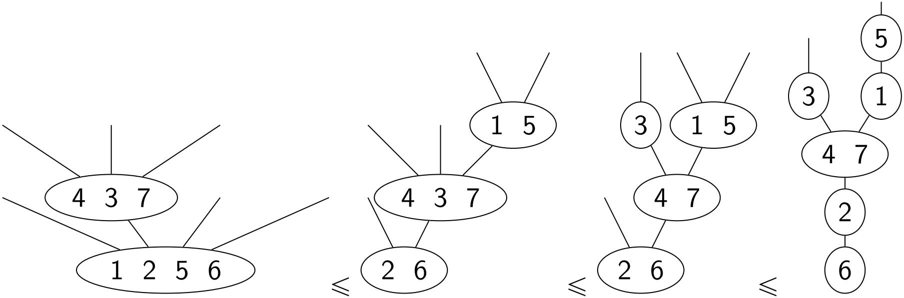
A $k$-parking tree on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

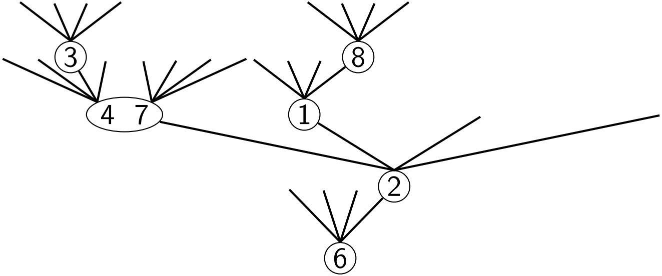
- $V \in \Pi_L$
- $v \in V$ has $k|v|$ children.




5
3
6
7
4
12


3
5
12467
⩽
5
6
3
4
127
⩽
7
5
6
3
4
12


4 3 7
1 2 5 6
⩽
1 5
4 3 7
2 6
⩽
3
1 5
4 7
2 6
⩽
5
3
1
4 7
2
6


3
8
4 7
1
2
6

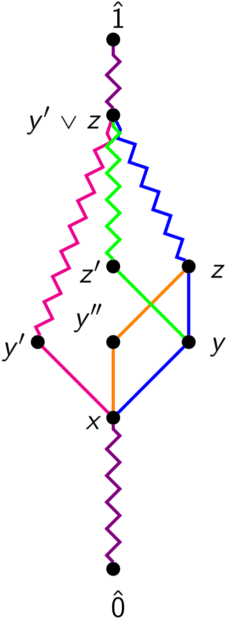
## Proposition (DO,Josuat-Vergès,Randazzo, 20+)

*Chains* $\phi_1 \leqslant \cdots \leqslant \phi_k$ *in* $\square_n$ *are in bijection with k-parking trees.*
*The number of chains* $\phi_1 \leqslant \cdots \leqslant \phi_k$ *in* $\square_n$ *where* $\mathrm{rk}(\phi_k) = \ell$ *is:*

$$\ell! \binom{kn}{\ell} S_2(n, \ell + 1).$$

Thank you !

# Shelling




## Lemma

*Let $x, y, y', z \in ?_n$ such that $x \lessdot y \lessdot z$, $x \lessdot y'$, and $y' \prec_x y$. Then:*

- *either there exists $y'' \in ?_n$ such that $x \lessdot y'' \lessdot z$ and $y'' \prec_x y$,*
- *or there exists $z' \in ?_n$ such that $y \lessdot z' \leqslant y' \vee z$ and $z' \prec_y z$.*