# Parking trees

(ArXiv : 2103.14468)

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11 mai 2021

## ♪ Outline ♪

1. *L'avventura* : Set and noncrossing partitions
2. *Michelle* : Parking functions
3. *Video killed the radio star*: Parking trees and species
4. *Blinding Lights* : Parking poset

# *L'avventura* : Set and noncrossing partitions

# Set partition and noncrossing partitions

**Definition**

A (set) partition of $E$ is $\pi = \{\pi_1, \ldots, \pi_k\}$ s.t. :

- $\pi_k \cap \pi_l \neq \emptyset \implies k = l$
- and $\bigcup_{i=1}^{k} \pi_i = E$.

$\Pi_E$ = set of partitions of $E$

Examples :

**Definition (Kreweras, 1972)**

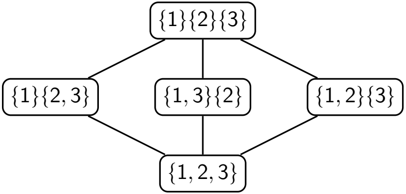
A partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of $\{1, \ldots, n\}$ is non-crossing iff

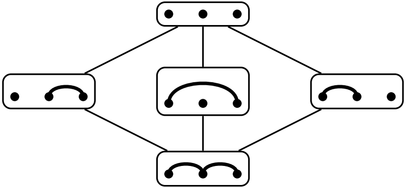
$$\begin{cases} a < b < c < d \\ a, c \in \pi_i \\ b, d \in \pi_j \end{cases} \implies i = j$$

$NC_n$ = set of non-crossing partitions of $\{1, \ldots, n\}$

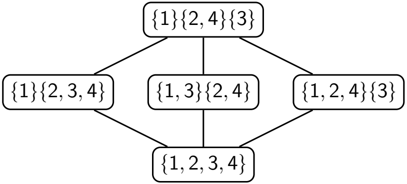
→ Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$

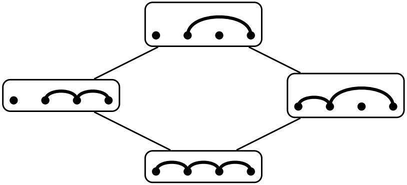
# Partitions and non-crossing partitions poset



# Partitions and non-crossing partitions poset



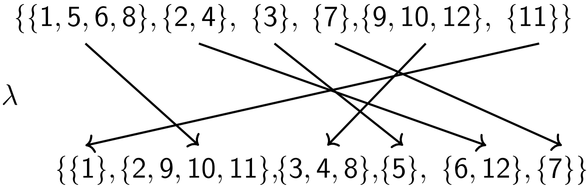



# Non-crossing 2-partitions

**Definition (Edelman, 1980)**

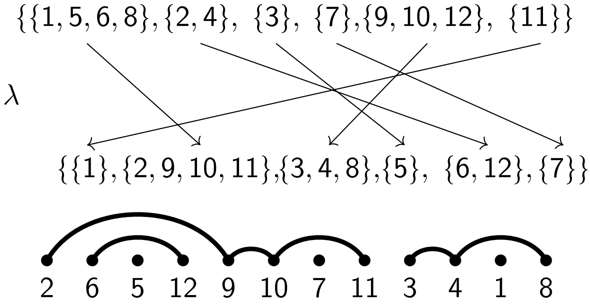
A n.c. 2-partition of $E$ is a triple $(\pi, \rho, \lambda)$ where:

- $\pi \in NC_{|E|}$ and $\rho \in \Pi_E$,
- $\lambda : \pi \hookrightarrow\!\!\!\!\rightarrow \rho$ s.t. $\forall B \in \pi$, $|\lambda(B)| = |B|$.



# First result : Bijection between 2NCP and labelled NCP

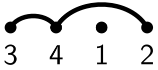
Labelled NCP : every element labelled so that labelling increases from left to right in the same part



# 2NCP poset

Covering relation in 2 : rearranging labels to respect the increasing condition

Example :




$\longrightarrow (n+1)^{n-1}$

*Michelle* : Parking functions

# Parking function [Konheim-Weiss, 1966]

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |

**Question :**

How to park 6 cars in 6 parking spaces ?

Easy answer : bijection between the parking spaces and cars.

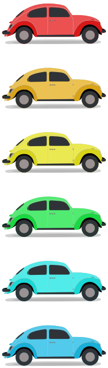
What if you pick at random for each car its place ? Can all cars park ?

$\rightarrow$ If yes, parking function !

# Parking function : examples and counter-examples

136524 - 122333 - 416114 - 153436

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |



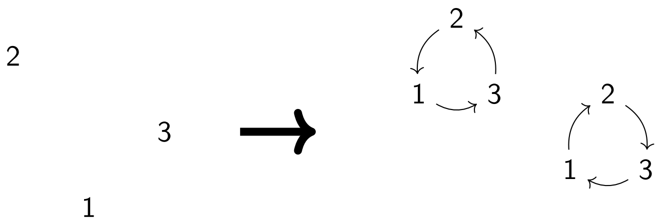
$f : \{1, \ldots, n\} \to \{1, \ldots, n\}$
s.t. $\bigcup_{j=1}^{i} |f^{-1}(j)| \geqslant i$

# *Video killed the radio star*: Parking trees and species

# Species [Joyal, 1980]

**Definition**

Species F : FinSet → FinSet which associates to a fin. set $I$, the fin. set F$(I)$, only depending on the cardinality of $I$.



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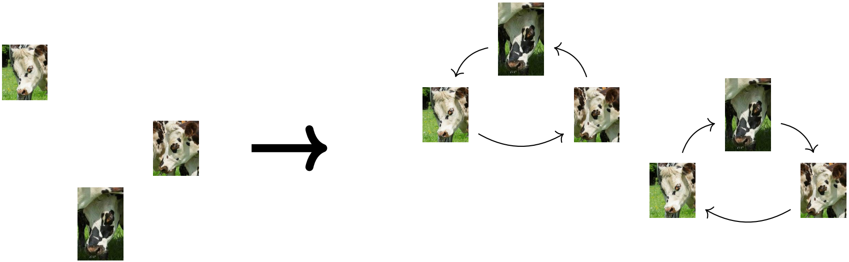


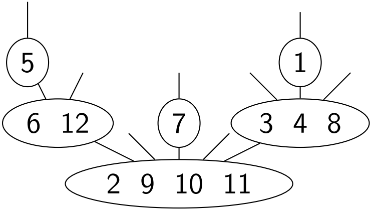
Species encodes the action of the symmetric group

# Parking trees

## Definition

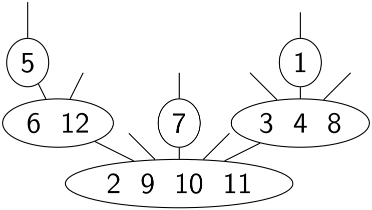
A parking tree on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

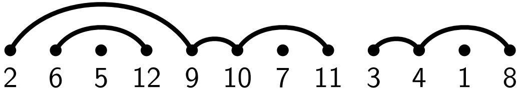
- $V \in \Pi_L$,
- $v \in V$ has $|v|$ children.




Why parking ?

# Bijection between 2NCP <-> parking trees



# Why do we need species ?

Let $F$ and $G$ be two species.

- $(F+G)(I) = F(I) \sqcup G(I)$,

- $(F \times G)(I) = \bigsqcup_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$.

## Definition

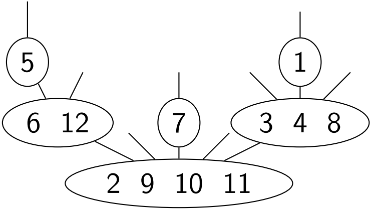
The cycle index series of a species $F$ is the formal power series in an infinite number of variables $\mathfrak{p} = (p_1, p_2, p_3, \ldots)$ defined by:

$$Z_F(\mathfrak{p}) = \sum_{n \geqslant 0} \frac{1}{n!} \left( \sum_{\sigma \in \mathfrak{S}_n} F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \ldots \right),$$

- with $F^\sigma = |\{x \in F(\{1, \ldots, n\}) | \sigma \cdot x = x\}|$
- and $\sigma$ has $\sigma_i$ cycles of length $i$.

Example :

# Back to parking trees

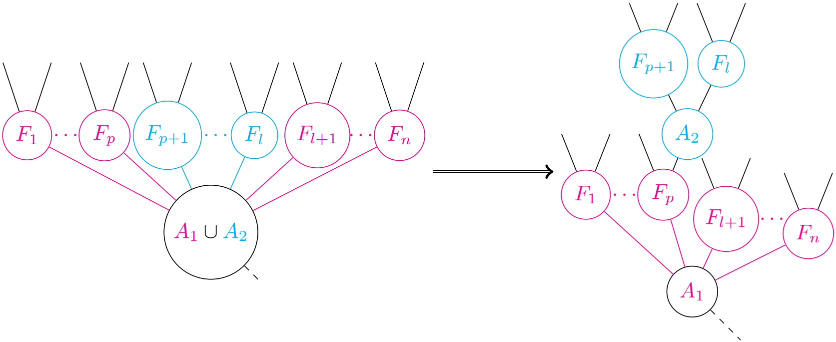


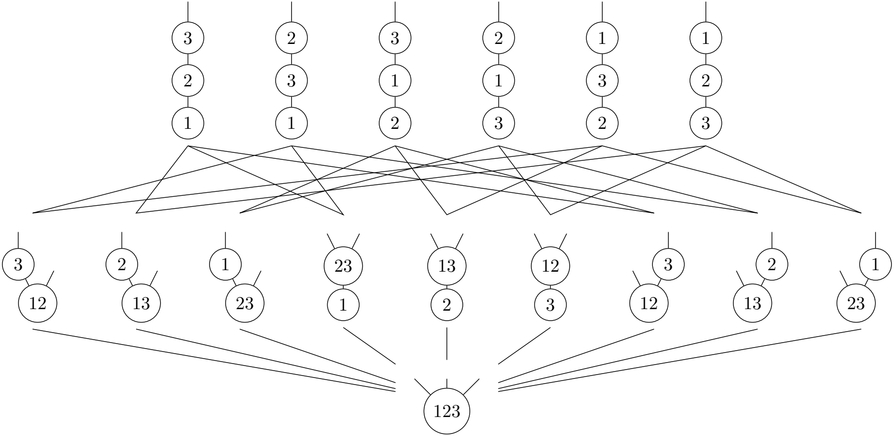

**Proposition (DO, Josuat-Vergès, Randazzo, 20+)**

$$\mathcal{P}_f = \sum_{p \geqslant 1} \mathcal{E}_p \times (1 + \mathcal{P}_f)^p$$

*Blinding Lights* : Parking poset

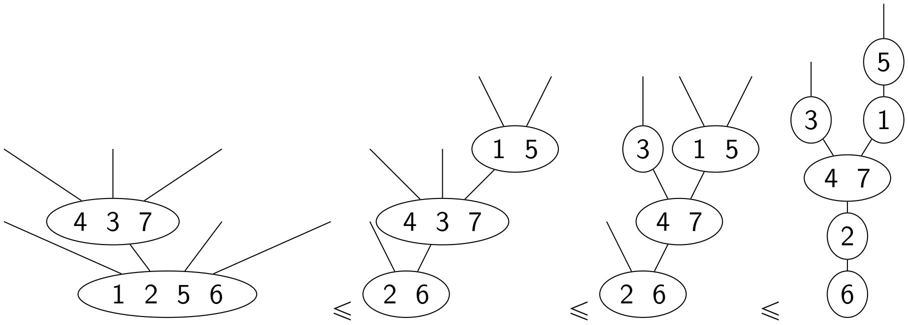
# Order on parking trees

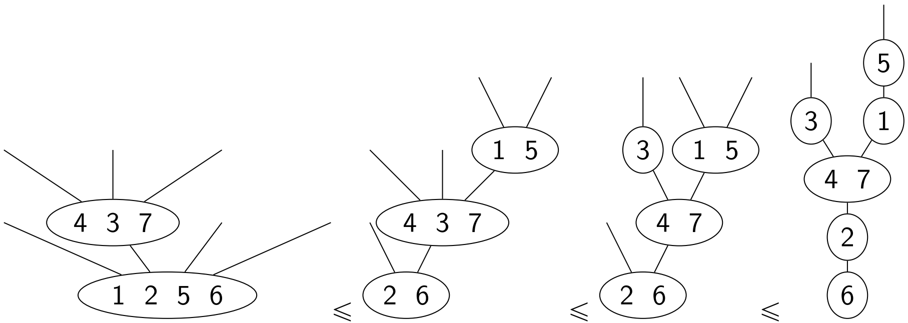



3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3
3 12
2 13
1 23
23 1
13 2
12 3
3 12
2 13
1 23
123

# Results

- This poset is a lattice
- When restricting right combs, get the face poset of the permutohedron
- New criterion to prove shelling !
- Enumeration of (weak) k-chains






**Proposition (DO, Josuat-Vergès, Randazzo, 20+)**

$$\mathcal{C}^{l}_{k,t} = \sum_{p \geqslant 1} \mathcal{C}^{l,p}_{k-1,t} \times \left(t\mathcal{C}^{l}_{k,t} + 1\right)^{p},$$

## Proposition (DO,Josuat-Vergès,Randazzo, 20+)

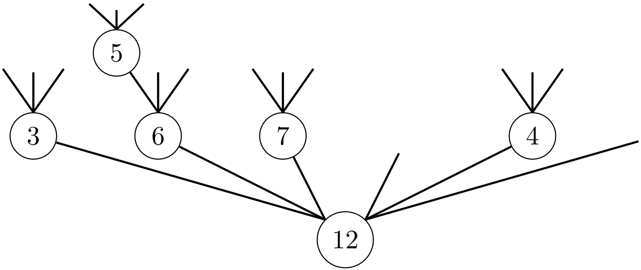
*Chains $\phi_1 \leqslant \cdots \leqslant \phi_k$ in ⍰$_n$ are in bijection with k-parking trees.*
*The number of chains $\phi_1 \leqslant \cdots \leqslant \phi_k$ in ⍰$_n$ where $\mathrm{rk}(\phi_k) = \ell$ is:*

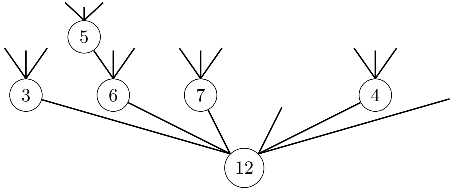
$$\ell! \binom{kn}{\ell} S_2(n, \ell + 1).$$

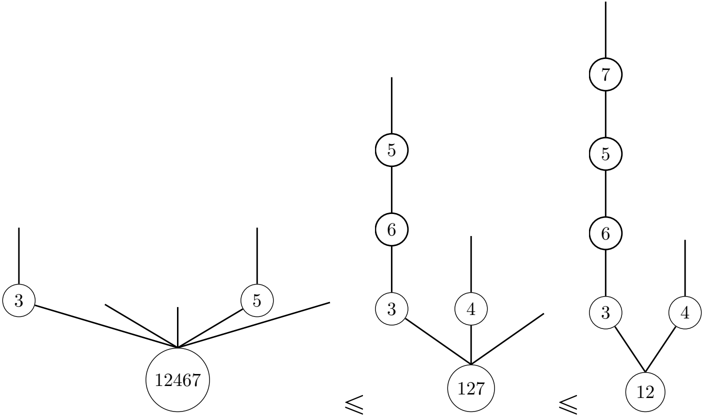
## Definition

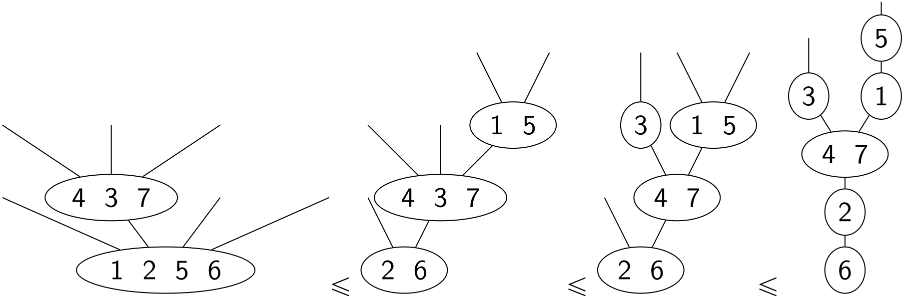
A *$k$-parking tree* on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

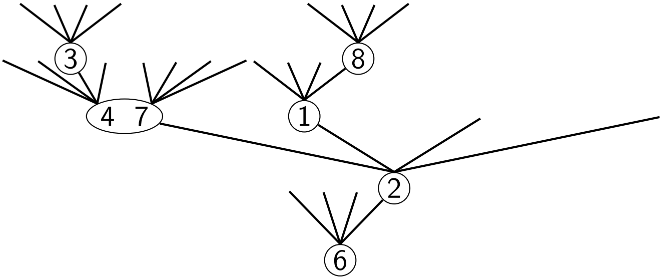
- $V \in \Pi_L$
- $v \in V$ has $k|v|$ children.




5
3
6
7
4
12


3
5
12467
⩽
5
6
3
4
127
⩽
7
5
6
3
4
12


4 3 7
1 2 5 6
≤
1 5
4 3 7
2 6
≤
3
1 5
4 7
2 6
≤
5
3
1
4 7
2
6


3
8
4 7
1
2
6

**Proposition (DO, Josuat-Vergès, Randazzo, 20+)**

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*The number of chains $\phi_1 \leqslant \cdots \leqslant \phi_k$ in ⍰$_n$ where $\mathrm{rk}(\phi_k) = \ell$ is:*

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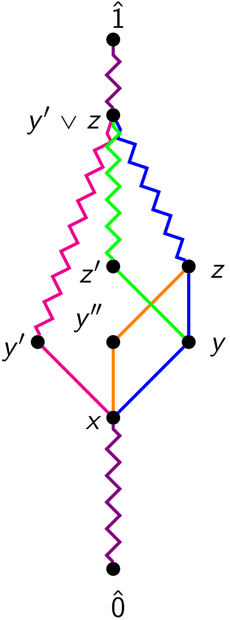
Thank you !

# Shelling




## Lemma

*Let* $x, y, y', z \in ⍰_n$ *such that* $x \lessdot y \lessdot z$, $x \lessdot y'$, *and* $y' \prec_x y$. *Then:*

- *either there exists* $y'' \in ⍰_n$ *such that* $x \lessdot y'' \lessdot z$ *and* $y'' \prec_x y$,
- *or there exists* $z' \in ⍰_n$ *such that* $y \lessdot z' \leqslant y' \vee z$ *and* $z' \prec_y z$.