Posets, incidence Hopf algebras and operads

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Outline

1. Posets and incidence Hopf algebra
2. Hypertrees
3. Operads and homology
4. Back to the cohomology of the hypertree posets
   joint work with Clément Dupont (IMAG)

Posets and incidence Hopf algebra

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Hypertrees

Outline

Operads and homology

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Recall from yesterday

Warning !

Today, we will consider the posets of hypertrees on $\{0, \dots, n\}$, denoted by HT_n with a slight abuse of the notation !

Proposition (McCullough–Miller, 1996)

HT_n *is Cohen-Macaulay.*

Proposition

The Möbius number of HT_n *is given by:*

$$\mu(\mathrm{HT}_n) = (-1)^n \frac{(2n-1)!}{n!}$$

Post-Lie operad [Vallette, 07 ; Munthe-Kaas–Wright, 08]

The underlying vector space PostLie(V) of post-Lie operad is spanned by Lie brackets of planar trees with nodes labeled by V. The substitution of a tree t inside a node v is given by the sum over all the way to graft each child of v to the right of a node of t (planar pre-Lie product).

Proposition

The vector space of n-ary operations of Post-Lie operad has dimension $\text{Post-Lie}(n) = \frac{(2n-1)!}{n!}$.

The post-Lie products $\lhd$ and $\{\ ;\ \}$ satisfy

- $\{\ ;\ \}$ is a Lie bracket

$$(x \lhd y) \lhd z - x \lhd (y \lhd z) - (x \lhd z) \lhd y + x \lhd (z \lhd y) = x \lhd \{y, z\}$$
$$\{x, y\} \lhd z = \{x \lhd z, y\} + \{x, y \lhd z\}$$

Back to the cohomology of the hypertree posets
joint work with Clément Dupont (IMAG)

Outline

Motivation : Partition posets and Lie operad

$$C_j(\Pi_n) = \mathbb{C}.\{\hat{0}_{\Pi_n} = \pi_{-1} < \ldots < \pi_{j+1} = \hat{1}_{\Pi_n} | \pi_l \in \Pi_n, \forall l \in [\![-1; j+1]\!]\}$$

Example : leveled cobar construction

Theorem (Fresse, 04)

The action of the symmetric group on the cohomology of the partition posets Π_n is given by

$$\tilde{H}_{n-1}(\Pi_n) = \mathrm{Lie}(n)^\vee \otimes sgn_n$$

where $\mathrm{Lie}(n)^\vee$ is the dual module of Lie.

To nested sets

Problem

There are no operadic structure on the leveled cobar construction, but there is one on the cobar construction !

Solution :

Forget about the levels !

This is what we obtain when we consider nested sets instead of chains !

Join and meet

Definition

The join of a subset S of a poset P is the supremum (least upper bound) of S, denoted by $\bigvee S$.

In the partition poset

$\{1,3\}\{2,4,6\}\{5\} \vee \{1,2,3\}\{4,6\}\{5\} = \{1,3\}\{2\}\{4,6\}\{5\}$.
More generally, the join of two partitions $p = \{p_1, \ldots, p_k\}$ and $q = \{q_1, \ldots, q_l\}$ is $\{p_i \cap q_j\}_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l}$

Definition

The meet of a subset S of a poset P is the infimum (greatest lower bound) of S, denoted by $\bigwedge S$.

In the partition poset

The meet of two partitions p and q is obtained by merging every parts p_1 and p_2 of p such that there is a part in q intersecting non trivially both.

Lattices

Definition

A join-semilattice is a poset that has a join for any nonempty finite subset.

The partition posets are join-semilattices.

Lemma

The cartesian product of join-semilattices is a join-semilattice.

Lemma

$$\mathrm{HT}_n^a = \prod_{v \in V(a)} \Pi_{\deg(v)}$$

Proposition

Every maximal interval HT_n^a in the hypertree posets is a join-semilattice.

Let us recall from the first lecture

Let $\pi \in \Pi_n$, $\pi = \{V_1, \ldots, V_k\}$

Lemma

The following isomorphisms hold:

$$[\pi, 1_{\Pi_n}] \simeq \prod_{i=1}^{k} \Pi_{|V_i|} \qquad [0_{\Pi_n}, \pi] \simeq \Pi_k$$

Building sets [De Concini–Procesi, 95 ; Feichtner–Müller, 05]

Consider $\mathcal{L}$ a finite join-semilattice. For any $S \subseteq \mathcal{L}$ and $x \in \mathcal{L}$, we write

$$S_{\geqslant x} = \{y \in S | y \geqslant x\}.$$

Definition

A building set is a subset $\mathcal{G}$ in $\mathcal{L}_{<\hat{1}}$ such that for any $x \in \mathcal{L}_{<\hat{1}}$ and $\max \mathcal{G}_{\geqslant x} = \{g_1, \ldots, g_k\}$,there is an isomorphism of posets

$$[x, \hat{1}] \simeq \prod_{i=1}^{k} [g_i, \hat{1}].$$

Example on partition posets:

The minimal building set for partition posets is the set of partitions whose part are all trivial but one.

Nested sets [De Concini–Procesi, 95 ; Feichtner–Müller, 05]

Consider a join-semilattice $\mathcal{L}$ and an associated building set $\mathcal{G}$.

Definition

A nested set is a subset S of $\mathcal{G}$ such that for any set of incomparable elements $x_1, \ldots, x_t$ in S ($t \geqslant 2$), the meet $x_1 \wedge \ldots \wedge x_t$ exists and does not belong to $\mathcal{G}$.

Example on partition posets:

Topological result

The $\mathcal{G}$-nested sets form an abstract simplicial complex, called the nested set complex.

Proposition (Feichtner–Müller, 05)

Consider a join-semilattice $\mathcal{L}$ and an associated building set $\mathcal{G}$. The associated nested set complex is homotopy equivalent to the order complex of the poset.

For partition posets

The cobar resolution (for the Commutative operad) corresponds to the cochain complex of the nested set complex associated with the minimal building set.

Let us adapt building sets to hypertree posets

Definition

A building set of HT_n is a subset $\mathcal{G}$ in $HT_n - \max(HT_n)$ such that for any $x \in HT_n$ and $\max \mathcal{G}_{\geqslant x} = \{g_1, \ldots, g_k\}$,there is an isomorphism of posets for every tree a in HT_n:

$$[x, a] \simeq \prod_{i=1}^{k} [g_i, a].$$

Example on hypertree posets:

The minimal building set for hypertree posets is the set of hypertrees whose edges are of size two but one.

The nested set complex of hypertrees

The nested sets of hypertrees are the following combinatorial objects:

Definition

A bitree is a pair (T, τ) of trees such that

- T is a (non planar) rooted reduced (no vertex of valency 2) tree with leaves labeled by $\{1, \ldots, n\}$
- τ is a (non planar oriented) tree whose vertices are labeled by $\{0, \ldots, n\}$ and whose root is 0
- for any internal vertex s in T, the restriction of τ to edges leaving the leaves above s is connected

Operadic composition

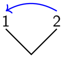
The operadic composition of a bitree b in a node v is as follows:

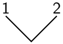
- the blue children of v are grafted to some nodes in b (pre-Lie type magmatic composition)
- the bottom tree of b is grafted at the place of the leaf v (usual magmatic composition)

Operadic structure on the cohomology of the nested set complex (aka. post-Lie !)

Let us consider the map

$$\text{Post-Lie} \xrightarrow{\phi} H^{\bullet}(HT_{\bullet})$$

$1 \lhd 2 \mapsto$ 

$\{1; 2\} \mapsto$ 

Theorem (DO–Dupont, 22+)

The map ϕ is an operad morphism. The cohomology of the hypertree poset can be endowed with an operadic structure. It is then isomorphic to the suspension of post-Lie operad.

Idea of the proof : Comb and twist

First, we choose a planar representation of the bitree with blue arrows to the left. The proof relies on two lemmas:

Lemma (Comb)

In the cohomology, any bitree is equivalent to the bitree obtained by exchanging the bottom tree with a comb-shaped tree

Lemma (Twist)

In the cohomology, any nodes which are not close relatives (parent-child or siblings) can be exchanged

We then prove that ϕ is a morphism of operad and is surjective.

Let us consider the map $a : \mathrm{HT}_n^0 \to \Pi_n$. We define

$$(\mathrm{HT})_{\leqslant\pi} := a^{-1}\left(\Pi_{\leqslant\pi}\right) \text{ and } (\mathrm{HT})_{\geqslant\pi} := a^{-1}\left(\Pi_{\geqslant\pi}\right).$$

Define the maps

$$\varphi : (\mathrm{HT})_{\leqslant\pi} \to \mathrm{HT}(\pi)$$

and

$$\psi : (\mathrm{HT})_{\geqslant\pi} \to \prod_{t\in\pi} \mathrm{HT}(t)$$

obtained respectively by contracting parts of π to an element and splitting the hypertree according to the parts of π.
The idea is to use these maps to define a composition:

$$C^\bullet\left(\mathrm{HT}(\pi)\right) \otimes \bigotimes_{T\in\pi} C^\bullet\left(\mathrm{HT}(T)\right) \simeq C^\bullet\left(\mathrm{HT}(\pi)\right) \otimes C^\bullet\left(\prod_{T\in\pi} \mathrm{HT}(T)\right)$$
$$\xrightarrow{\phi^*\otimes\psi^*} C^\bullet\left(\mathrm{HT}_{\leqslant\pi}\right) \otimes C^\bullet\left(\mathrm{HT}_{\geqslant\pi}\right) \to C^\bullet\left(\mathrm{HT}_n\right)$$

What about pre-Lie ?

Work in progress

- We hope to recover pre-Lie as a left post-Lie module.
- Hypertree poset has the same cohomology as the poset of partitions decorated by Operad ComTrias [Vallette, 07] : both posets are two projections of the poset of multi-pointed hypertrees (whose cohomology group is post-Lie operad).
- This fits in a more general setting, which applies to examples of families of poset (F_n) endowed with a "good" projection $a : F_n \to \Pi_n$.

References

- Koszul duality of operads and homology of partition posets, B. Fresse, *"Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory", Contemp. Math. 346, Amer. Math. Soc., 115-215* (2004)
- On the topology of nested set complexes, E. Feichtner–I. Müller, *Proc. Am. Math. Soc. 133, No. 4, 999-1006* (2005)

Thank you very much for your attention to this four lectures !

as I have checked carefully with the fake mistakes in these slides !