Posets, incidence Hopf algebras and operads

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Outline

1. Posets and incidence Hopf algebra
2. Hypertrees
3. Operads and homology
4. Back to the homology of the hypertree posets

Posets and incidence Hopf algebra

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Hypertrees

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Möbius number of the hypertree posets

Proposition (McCammond-Meier, 2004)

The Möbius number of $\widehat{\mathrm{HT}}_n$ *is given by:*

$$\mu(\widehat{\mathrm{HT}}_n) = (-1)^{n-1}(n-1)^{n-2}$$

Proposition

The Möbius number of HT_n *is given by:*

$$\mu(\mathrm{HT}_n) = (-1)^n \frac{(2n-3)!}{(n-1)!}$$

Operads and homology

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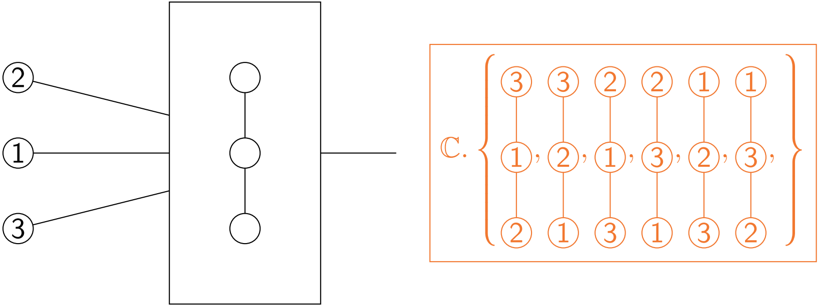
What are species?

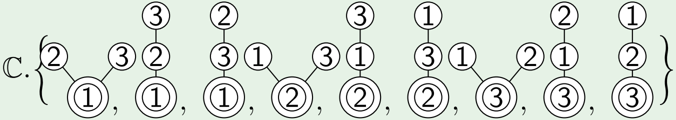
Definition (Joyal, 80s)

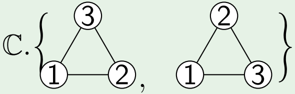
A species F is a functor from Bij to Vect. To a finite set S, the species F associates a vector space $\mathrm{F}(S)$ independent from the nature of S.

Species = Construction plan, such that the vector space obtained is invariant by relabeling



Examples of species

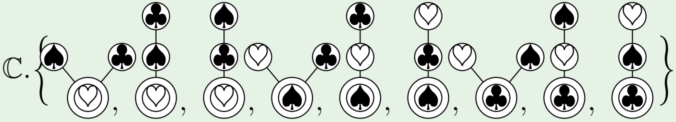
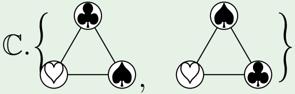
- $\mathbb{C}.\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists $\mathbb{L}$ on $\{1,2,3\}$)
- $\mathbb{C}.\{\{1,2,3\}\}$ (species of non-empty sets $\mathbb{E}^+$)
- $\mathbb{C}.\{\{1\},\{2\},\{3\}\}$ (species of pointed sets $\mathbb{E}^\bullet$)
- 

(Species of Cayley trees $\mathbb{T}$)
- 

(Species of cycles)

These sets are the image by species of the set $\{1,2,3\}$.

Examples of species

- $\mathbb{C}.\{(\heartsuit,\spadesuit,\clubsuit),(\heartsuit,\clubsuit,\spadesuit),(\spadesuit,\heartsuit,\clubsuit),(\spadesuit,\clubsuit,\heartsuit),(\clubsuit,\heartsuit,\spadesuit),(\clubsuit,\spadesuit,\heartsuit)\}$ (Species of lists $\mathbb{L}$ on $\{\clubsuit,\heartsuit,\spadesuit\}$)
- $\mathbb{C}.\{\{\heartsuit,\spadesuit,\clubsuit\}\}$ (Species of non-empty sets $\mathbb{E}^+$)
- $\mathbb{C}.\{\{\heartsuit\},\{\spadesuit\},\{\clubsuit\}\}$ (Species of pointed sets $\mathbb{E}^\bullet$)
-  (Species of rooted trees $\mathbb{T}$)
-  (Species of cycles)

These sets are the image by species of the set $\{\clubsuit,\heartsuit,\spadesuit\}$.

Substitution of species

Proposition

Let F and G be two species. Let us define:

$$(F \circ G)(S) = \bigoplus_{\pi \in \mathcal{P}(S)} F(\pi) \otimes \bigotimes_{J \in \pi} G(J),$$

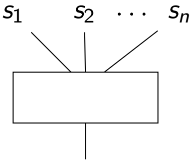
where $\mathcal{P}(S)$ runs on the set of partitions of S.

Example

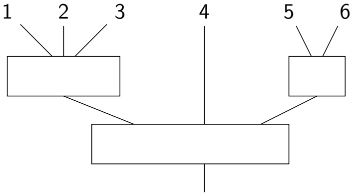
Operads

An operad $\mathcal{O}$ is

- a species $\mathcal{O}$
- with an associative composition



$$\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$$



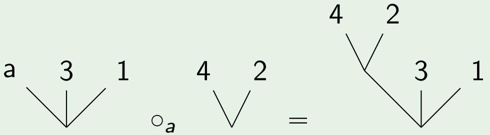
- and a unit $i : I \to \mathcal{O}$, where I is the singleton species ($I(S) = \delta_{|S|=1}\mathbb{C}$).
- To each kind of algebra is associated an operad.

Free operad

Let M be $\mathfrak{S}$-module. The free operad over M is the operad whose underlying species associate to any finite set V the set of rooted trees whose leaves are labeled by V and whose inner vertices are labeled by an element of M, with sustitution given by grafting on leaves.

Mag operad

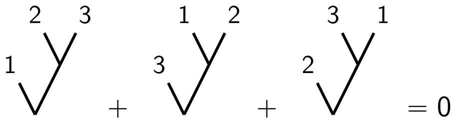
When $M = \mathbb{C}.\{(1,2),(2,1)\}$, the free operad is called Magmatic operad. The species $\mathcal{M}ag(V)$ is the species of planar binary trees with leaves labeled by V.



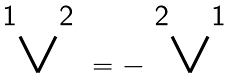
Any operad can be described as a quotient of a free operad.

Lie operad

Lie operad encodes Lie algebra. Its underlying vector space is obtained as a quotient of the Magmatic operad's vector spaces with the Jacobi relations



and the anti-symmetry



Proposition

The vector space of n-ary operations of Lie operad has dimension $\mathrm{Lie}(n) = (n-1)!$ *(comb).*

Pre-Lie operad [Chapoton–Livernet, 00; Dzhumadil'daev–Löfwall, 02]

The underlying vector space PreLie(V) of pre-Lie operad is spanned by Cayley trees with nodes labeled by V. The substitution of a tree t inside a node v is given by the sum over all the ways to graft each child of v on a node of t.

Proposition

The vector space of n-ary operations of Pre-Lie operad has dimension $\mathrm{Pre\text{-}Lie}(n) = n^{n-1}$.

The pre-Lie product $\leftharpoondown$ satisfy the following relation for any elements x, y and z:

$$(x \leftharpoondown y) \leftharpoondown z - x \leftharpoondown (y \leftharpoondown z) = (x \leftharpoondown z) \leftharpoondown y - x \leftharpoondown (z \leftharpoondown y).$$

Post-Lie operad [Vallette, 07 ; Munthe-Kaas–Wright, 08]

The underlying vector space PostLie(V) of post-Lie operad is spanned by Lie brackets of planar trees with nodes labeled by V. The substitution of a tree t inside a node v is given by the sum over all the way to graft each child of v to the right of a node of t (planar pre-Lie product).

Proposition

The vector space of n-ary operations of Post-Lie operad has dimension $\text{Post-Lie}(n) = \frac{(2n-1)!}{n!}$.

The post-Lie products $\lhd$ and $\{\ ;\ \}$ satisfy

- $\{\ ;\ \}$ is a Lie bracket

$$(x \lhd y) \lhd z - x \lhd (y \lhd z) - (x \lhd z) \lhd y + x \lhd (z \lhd y) = x \lhd [y, z]$$

$$\{x, y\} \lhd z = \{x \lhd z, y\} + \{x, y \lhd z\}$$

Möbius number of the poset = Euler characteristic

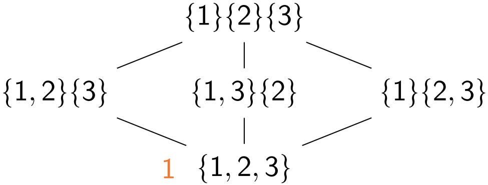
Definition

For any poset P, the Möbius function is defined on any interval $x \leqslant_P y$ by:

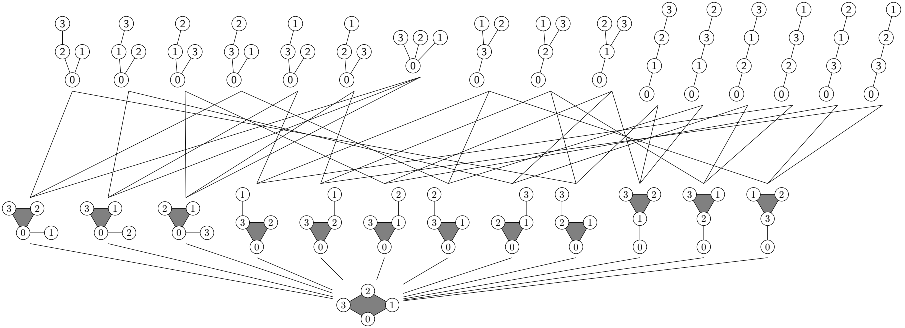
$$\mu(x,x) = 1, \qquad \forall x \in P$$

$$\mu(x,y) = -\sum_{x \leqslant z < y} \mu(x,z), \qquad \forall x < y \in P.$$

If P is bounded, its Möbius number is $\mu(P) := \mu(\hat{0}, \hat{1})$.



Möbius number of the hypertree poset



First exercice

Compute the Möbius number of the boolean posets. Check that it is consistent with the results presented in the first lesson.

(Co)homology of a poset

Let P be a poset.

$C_j(P) = \mathbb{C}$-vector space of j-chains $x_0 < x_1 < \ldots < x_j$ of P, with $C_{-1}(P) = \mathbb{C}.e$

For $j \geqslant 0$, let us define the differential $\partial_j : C_j(P) \to C_{j+1}(P)$ by:

$$\partial(x_0 < x_1 < \ldots < x_j) = \sum_{i=1}^{j+1} (-1)^i (x_0 < x_1 < \ldots < x_{i-1} < x < x_i < \ldots < x_j).$$

We have $\partial_j \partial_{j-1} = 0$.
The jth cohomology group is then defined, for any $j \geqslant 0$, by:

$$\tilde{H}^j(P) = \ker \partial_j / \operatorname{im} \partial_{j-1}.$$

Back to the Möbius numbers

Let P be a poset.

$$\mu(P) = \sum_{k=-1}^{\infty} (-1)^k \dim(C_k(P)) = \sum_{k=0}^{\infty} (-1)^k \dim\left(\tilde{H}^k(P)\right)$$

Cohen-Macaulay posets

Theorem (Björner, 1980)

The partition poset Π_n is Cohen-Macaulay (even EL-shellable): all its cohomology group vanish but its top one.

$\rightarrow$ In this case, the Möbius number gives, up to a sign, the dimension of the unique non trivial cohomology group.

Cohen-Macaulay posets

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→ In this case, the Möbius number gives, up to a sign, the dimension of the unique non trivial cohomology group.

Theorem (Hanlon, 81; Stanley, 82 ; Joyal, 85; Fresse, 04)

The action of the symmetric group on the cohomology of the partition posets Π_n is (nearly) given by:

$$\mathrm{Lie}(n) = \bigoplus_{\sigma \in \mathfrak{S}_n} \mathbb{C}.[\dots[\sigma(1), \sigma(2), \dots, \sigma(n)]\dots]/(\textit{anti-sym.} + \textit{rel. de Jacobi}),$$

where $[\dots[\dots]\dots]$ stands for the sum of all possible parenthesizing with Lie brackets of a word of size n.

HT_n is Cohen-Macaulay

Proposition (McCullough–Miller, 1996)

$\widehat{\mathrm{HT}}_n$ *and* HT_n *are Cohen-Macaulay.*

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References

- Combinatorial species and tree-like structures, F. Bergeron, G. Labelle and P. Leroux
- Algebraic operads, J.-L. Loday et B. Vallette
- Poset topology, M. Wachs

Back to the homology of the hypertree posets

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