Posets, incidence Hopf algebras and operads

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Outline

Posets and incidence Hopf algebra (Recall from yesterday)

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1. Posets and incidence Hopf algebra (Recall from yesterday)
2. Hypertrees
3. Operads and homology
4. Back to the homology of the hypertree posets

Coproduct of the algebra

Given $\mathbb{C}$ a commutative ring with unit, define $\mathcal{C} := \mathbb{C}.\mathcal{F}_P / \sim$, the free $\mathbb{C}$-module on the quotient $\mathcal{F}_P$ by isomorphism classes of posets.
$\mathcal{C}$ is endowed with the coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and the counit $\epsilon : \mathcal{C} \to \mathbb{C}$ defined by:

$$\Delta(P) = \sum_{x \in P} [0_P; x] \otimes [x, 1_P]$$

$$\epsilon(P) = \delta_{|P|=1}$$

Theorem (Schmitt)

$(\mathcal{C}, \Delta, \epsilon, \times, \nu, S)$ is a Hopf algebra.

Incidence Hopf algebra of the poset of partitions

Let $\pi \in \Pi_n$, $\pi = \{V_1, \dots, V_k\}$

Lemma

The following isomorphisms hold:

$$[\pi, 1_{\Pi_n}] \simeq \prod_{i=1}^{k} \Pi_{|V_k|} \qquad [0_{\Pi_n}, \pi] \simeq \Pi_k$$

The coproduct is given by:

$$\Delta\left(\frac{\Pi_n}{n!}\right) = \sum_{k=1}^{n} \sum_{(j_1,\dots,j_n)\in\mathbb{N}, \sum_{i=1}^{n} j_i = k, \sum_{i=1}^{n} i j_i = n} \binom{k}{j_1, \dots, j_n} \prod_{i=1}^{n} \left(\frac{\Pi_i}{i!}\right)^{j_i} \otimes \frac{\Pi_k}{k!}.$$

Incidence Hopf algebra of the boolean lattice

Let $V \in B_n$, $V = \{i_1, \dots, i_k\}$

Lemma

The following isomorphisms hold:

$$[V, \{1, \dots, n\}] \simeq B_{n-k} \qquad [\varnothing, V] \simeq B_k$$

The coproduct is given by:

$$\Delta\left(\frac{B_n}{n!}\right) = \sum_{k=0}^{n} \frac{B_k}{k!} \otimes \frac{B_{n-k}}{(n-k)!}.$$

Character of an incidence Hopf algebra

Consider the vector space of characters $\mathcal{H}^* = \mathrm{Hom}(\mathcal{H}, \mathbb{C})$ on an incidence Hopf algebra $\mathcal{H}$.
The convolution of two characters ϕ and ψ is given by:

$$\boxed{\phi * \psi = \sum \phi(P_{(1)})\psi(P_{(2)})}$$

where $\Delta(P) = \sum P_{(1)} \otimes P_{(2)}$.

On the partition and boolean posets

The vector space of characters on the incidence Hopf algebra of the partition posets corresponds to exponential generating functions (with the substitution) via $\phi \mapsto \sum_{n\geqslant 1} \frac{\phi(\Pi_n)}{n!} t^n$. The vector space of characters on the incidence Hopf algebra of the boolean posets corresponds to exponential generating functions (with the multiplication) via $\phi \mapsto \sum_{n\geqslant 0} \frac{\phi(B_n)}{n!} t^n$.

Some basic characters

Let us consider the character

$$\xi : \Pi_n \mapsto 1.$$

and let μ be its inverse for the convolution product.

For subsets

We have $\xi(t) = \sum_{n\geqslant 0} \xi(B_n)\frac{t^n}{n!} = \exp(t)$ and
$\mu(t) = \exp(-t) = \sum_{n\geqslant 0}(-1)^n \frac{t^n}{n!}$.

For partitions

We have $\xi(t) = \sum_{n\geqslant 1} \frac{\xi(\Pi_n)}{n!} t^n = \sum_{n\geqslant 1} \frac{1}{n!} t^n = \exp(t) - 1$ and
$\mu(t) = \ln(1+t) = \sum_{n\geqslant 1}(-1)^{n-1}(n-1)!\frac{t^n}{n!}$

Hypertrees

Outline

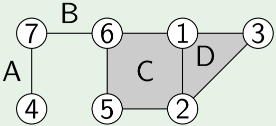
Hypergraphs

Definition (Berge)

A hypergraph (on a set V) is an ordered pair (V, E) where:

- V is a finite set (vertices)
- E is a collection of subsets of cardinality at least two of elements of V (edges).

Example of a hypergraph on $[1;7]$



Walk on a hypergraph

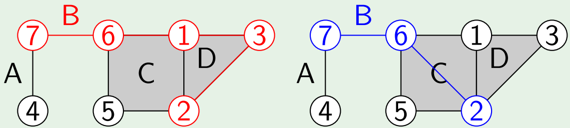
Definition

Let $H = (V, E)$ be a hypergraph.
A walk from a vertex or an edge d to a vertex or an edge f in H is an alternating sequence of vertices and edges beginning by d and ending by f:

$$(d, \ldots, e_i, v_i, e_{i+1}, \ldots, f)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.
The length of a walk is the number of edges and vertices in the walk.

Examples of walks



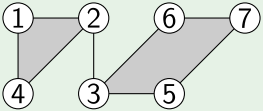
Hypertrees

Definition

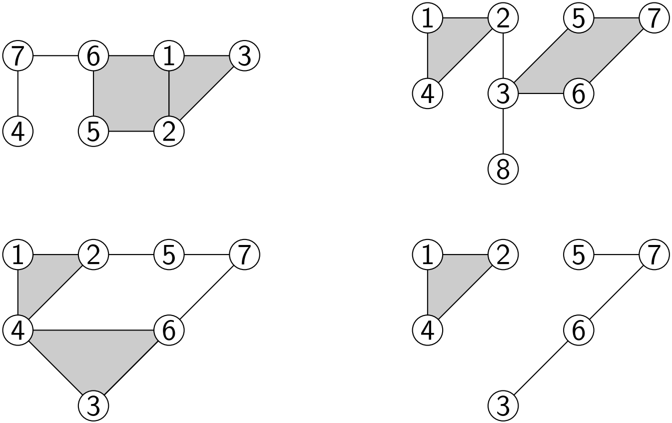
A hypertree is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

- there exists a walk from v to w in H with distinct edges e_i, (H is connected),
- and this walk is unique, (H has no cycles).

Example of a hypertree



First exercice : Which one(s) is/are a hypertree(s) ?



Numerology [Kalikow, 1999 ; Smith–Warme, 1998]

The number of hypertrees on n vertices is given by:

$$|\,\mathrm{HT}_n\,| = \sum_{k=1}^{n} n^{k-1} S(n-1, k)$$

A030019 Number of labeled spanning trees in the complete hypergraph on n vertices (all hyperedges having cardinality 2 or greater). 86

1, 1, 1, 4, 29, 311, 4447, 79745, 1722681, 43578820, 1264185051, 41381702275, 1509114454597, 60681141052273, 2667370764248023, 127258109992533616, 6549338612837162225, 361680134713529977507, 21333858798449021030515, 1338681172839439064846881 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,4

COMMENTS
Equivalently, this is the number of "hypertrees" on n labeled nodes, i.e. connected hypergraphs that have no cycles, assuming that each edge contains at least two vertices. - Don Knuth, Jan 26 2008. See A134954 for hyperforests.
Also number of labeled connected graphs where every block is a complete graph (cf. A035053).
Let H = (V,E) be the complete hypergraph on N labeled vertices (all edges having cardinality 2 or greater). Let e in E and K = |e|. Then the number of distinct spanning trees of H that contain edge e is g(N,K) = K * E[X_N^{N-K}] / N and the K=1 case gives this sequence. Clearly there is some deep structural connection between spanning trees in hypergraphs and Poisson moments.

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Warren D. Smith and David Warme, Paper in preparation, 2002.

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Alois P. Heinz, Table of n, a(n) for n = 0..370 (first 101 terms from T. D. Noe)
Ayomikun Adeniran and Catherine Yan, Gončarov Polynomials in Partition Lattices and Exponential Families, arXiv:1907.07814 [math.CO], 2019.
Ronald Bacher, On the enumeration of labelled hypertrees and of labelled bipartite trees, arXiv:1102.2708v1 [math.CO], 2011.
Maryam Bahrani and Jérémie Lumbroso, Enumerations, Forbidden Subgraph Characterizations, and the Split-Decomposition, arXiv:1608.01465 [math.CO]

The hypertree poset

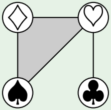
Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

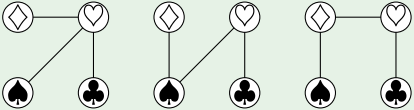
$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

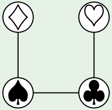
Example with hypertrees on four vertices



$\preceq$



but not



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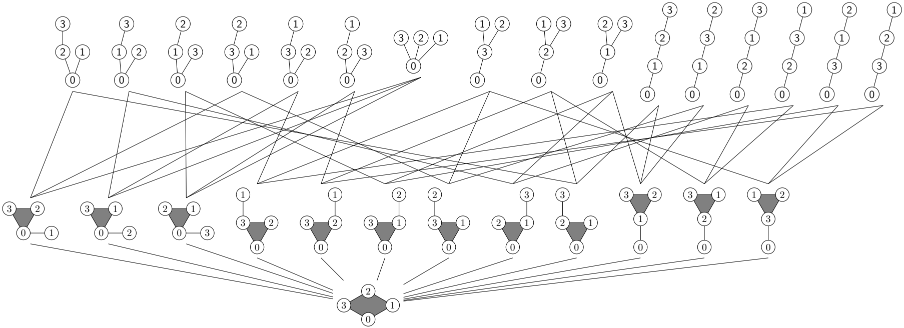

- $\widehat{\mathsf{HT}_\mathsf{n}}$ = augmented hypertree poset on $[\mathbf{0}, n]$.
- HT_n = hypertree poset on $[\mathbf{0}, n]$.
- For a a tree in HT_n, $\mathsf{HT}_\mathsf{n}^\mathsf{a}$ = maximal interval in hypertree poset on $[\mathbf{0}, n]$ between $\hat{0}$ and a.

Incidence Hopf algebra of the hypertree posets

Main question

What is the shape of an interval in HT_n ?

Proposition (McCammond–Meier, 2004)

Let H be a hypertree on n vertices and a be a tree such that $H \leqslant a$. The following isomorphisms hold:

$$[0_{\mathrm{HT}_n}, H] \simeq \prod_{v \in V(H)} \Pi_{\deg(v)} \qquad [H, a] \simeq \prod_{e \in E(H)} \mathrm{HT}_e^{a_{|e}}.$$

In particular, $\mathrm{HT}_n^a = \prod_{v \in V(a)} \Pi_{\deg(v)}$.

Möbius number of the hypertree posets

Proposition (McCammond-Meier, 2004)

The Möbius number of $\widehat{\mathrm{HT}}_n$ *is given by:*

$$\mu(\widehat{\mathrm{HT}}_n) = (-1)^{n-1}(n-1)^{n-2}$$

Proposition

The Möbius number of HT_n *is given by:*

$$\mu(\mathrm{HT}_n) = (-1)^n \frac{(2n-3)!}{(n-1)!}$$

$\frac{(2n-3)!}{(n-1)!}$?

A006963	Number of planar embedded labeled trees with n nodes: (2n-3)!/(n-1)! for n >= 2, a(1) = 1. (Formerly M3076)	28

1, 1, 3, 20, 210, 3024, 55440, 1235520, 32432400, 980179200, 33522128640, 1279935820800, 53970627110400, 2490952020480000, 124903451312640000, 6761440164390912000, 393008709555221760000, 24412776311194951680000, 1613955767240110694400000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS
For n>1: central terms of the triangle in A173333; cf. A001761, A001813. - Reinhard Zumkeller, Feb 19 2010
Can be obtained from the Vandermonde permanent of the first n positive integers; see A093883. - Clark Kimberling, Jan 02 2012
All trees can be embedded in the plane, but "planar embedded" means that orientation matters but rotation doesn't. For example, the n-star with n-1 edges has n! ways to label it, but rotation removes a factor of n-1. Another example, the n-path has n! ways to label it, but rotation removes a factor of 2. - Michael Somos, Aug 19 2014

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N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

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Robert Coquereaux and Jean-Bernard Zuber, Maps, immersions and permutations, Journal of Knot Theory and Its Ramifications, Vol. 25, No. 8 (2016), 1650047; arXiv preprint, arXiv:1507.03163 [math.CO], 2015-2016.
INRIA Algorithms Project, Encyclopedia of Combinatorial Structures 109.
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Pierre Leroux and Brahim Miloudi, Généralisations de la formule d'Otter, Ann. Sci.

has n! ways to label it, but rotation removes a factor of n-1. Another example, the n-path has n! ways to label it, but rotation removes a factor of 2. - Michael Somos, Aug 19 2014

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Pierre Leroux and Brahim Miloudi, Généralisations de la formule d'Otter, Ann. Sci. Math. Québec, Vol. 16, No. 1 (1992), pp. 53-80. (Annotated scanned copy)
J. W. Moon, Counting Labelled Trees, Issue 1 of Canadian mathematical monographs, Canadian Mathematical Congress, 1970.
Ran J. Tessler, A Cayley-type identity for trees, arXiv:1809.00001 [math.CO], 2018.
Index entries for sequences related to trees.

FORMULA

E.g.f. for a(n+1), n >= 1, log(c(x)); c(x) = g.f. for Catalan numbers A000108. - Wolfdieter Lang

Integral representation as n-th moment of a positive function on a positive half-

Reference(s)

- The hypertree poset and the l^2-Betti number of the motion group of the trivial link, J. McCammond and J. Meier, *Matematische Annalen, N°328, 633-652* (2004)

Operads and homology

Outline

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