Posets, incidence Hopf algebras and operads

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Outline

Posets and incidence Hopf algebra (Recall from yesterday)

Outline

Coproduct of the algebra

Given $\mathbb{C}$ a commutative ring with unit, define $\mathcal{C} := \mathbb{C}.\mathcal{F}_P / \sim$, the free $\mathbb{C}$-module on the quotient $\mathcal{F}_P$ by isomorphism classes of posets.
$\mathcal{C}$ is endowed with the coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and the counit $\epsilon : \mathcal{C} \to \mathbb{C}$ defined by:

$$\Delta(P) = \sum_{x \in P} [0_P; x] \otimes [x, 1_P]$$

$$\epsilon(P) = \delta_{|P|=1}$$

Theorem (Schmitt)

$(\mathcal{C}, \Delta, \epsilon, \times, \nu, S)$ is a Hopf algebra.

Incidence Hopf algebra of the poset of partitions

Let $\pi \in \Pi_n$, $\pi = \{V_1, \ldots, V_k\}$

Lemma

The following isomorphisms hold:

$$[\pi, 1_{\Pi_n}] \simeq \prod_{i=1}^{k} \Pi_{|V_k|} \qquad [0_{\Pi_n}, \pi] \simeq \Pi_k$$

The coproduct is given by:

$$\Delta\left(\frac{\Pi_n}{n!}\right) = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_n)\in\mathbb{N}, \sum_{i=1}^{n} j_i = k, \sum_{i=1}^{n} i j_i = n} \binom{k}{j_1, \ldots, j_n} \prod_{i=1}^{n} \left(\frac{\Pi_i}{i!}\right)^{j_i} \otimes \frac{\Pi_k}{k!}.$$

Incidence Hopf algebra of the boolean lattice

Let $V \in B_n$, $V = \{i_1, \dots, i_k\}$

Lemma

The following isomorphisms hold:

$$[V, \{1, \dots, n\}] \simeq B_{n-k} \qquad [\emptyset, V] \simeq B_k$$

The coproduct is given by:

$$\Delta\left(\frac{B_n}{n!}\right) = \sum_{k=0}^{n} \frac{B_k}{k!} \otimes \frac{B_{n-k}}{(n-k)!}.$$

Character of an incidence Hopf algebra

Consider the vector space of characters $\mathcal{H}^* = \mathrm{Hom}(\mathcal{H}, \mathbb{C})$ on an incidence Hopf algebra $\mathcal{H}$.
The convolution of two characters ϕ and ψ is given by:

$$\phi * \psi = \sum \phi(P_{(1)})\psi(P_{(2)})$$

where $\Delta(P) = \sum P_{(1)} \otimes P_{(2)}$.

On the partition and boolean posets

The vector space of characters on the incidence Hopf algebra of the partition posets corresponds to exponential generating functions (with the substitution) via $\phi \mapsto \sum_{n \geqslant 1} \frac{\phi(\Pi_n)}{n!} t^n$. The vector space of characters on the incidence Hopf algebra of the boolean posets corresponds to exponential generating functions (with the multiplication) via $\phi \mapsto \sum_{n \geqslant 0} \frac{\phi(B_n)}{n!} t^n$.

Some basic characters

Let us consider the character

$$\xi : \Pi_n \mapsto 1.$$

and let μ be its inverse for the convolution product.

For subsets

We have $\xi(t) = \sum_{n\geqslant 0} \xi(B_n)\frac{t^n}{n!} = \exp(t)$ and $\mu(t) = \exp(-t) = \sum_{n\geqslant 0}(-1)^n \frac{t^n}{n!}$.

For partitions

We have $\xi(t) = \sum_{n\geqslant 1} \frac{\xi(\Pi_n)}{n!} t^n = \sum_{n\geqslant 1} \frac{1}{n!} t^n = \exp(t) - 1$ and $\mu(t) = \ln(1+t) = \sum_{n\geqslant 1}(-1)^{n-1}(n-1)!\frac{t^n}{n!}$

Lie(n)

used later on

Hypertrees

Outline

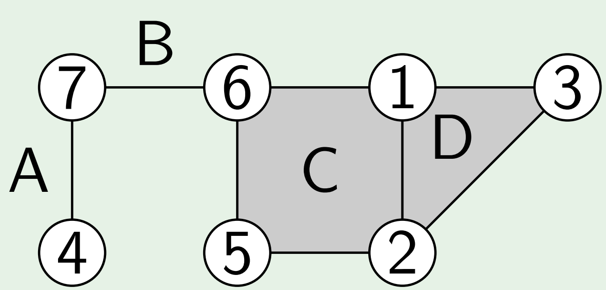
Hypergraphs

Definition (Berge)

A hypergraph (on a set V) is an ordered pair (V, E) where:

- V is a finite set (vertices)
- E is a collection of subsets of cardinality at least two of elements of V (edges).

Example of a hypergraph on $[1; 7]$



Walk on a hypergraph

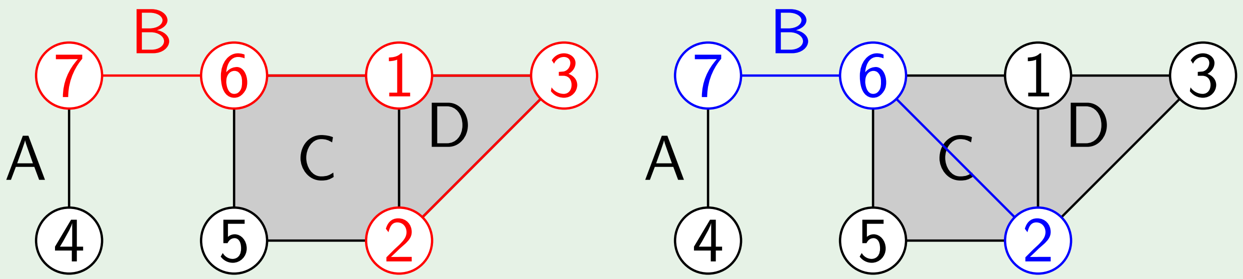
Definition

Let $H = (V, E)$ be a hypergraph.
A walk from a vertex or an edge d to a vertex or an edge f in H is an alternating sequence of vertices and edges beginning by d and ending by f:

$$(d, \ldots, e_i, v_i, e_{i+1}, \ldots, f)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.
The length of a walk is the number of edges and vertices in the walk.

Examples of walks



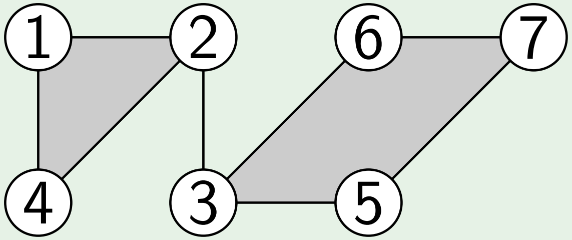
Hypertrees

Definition

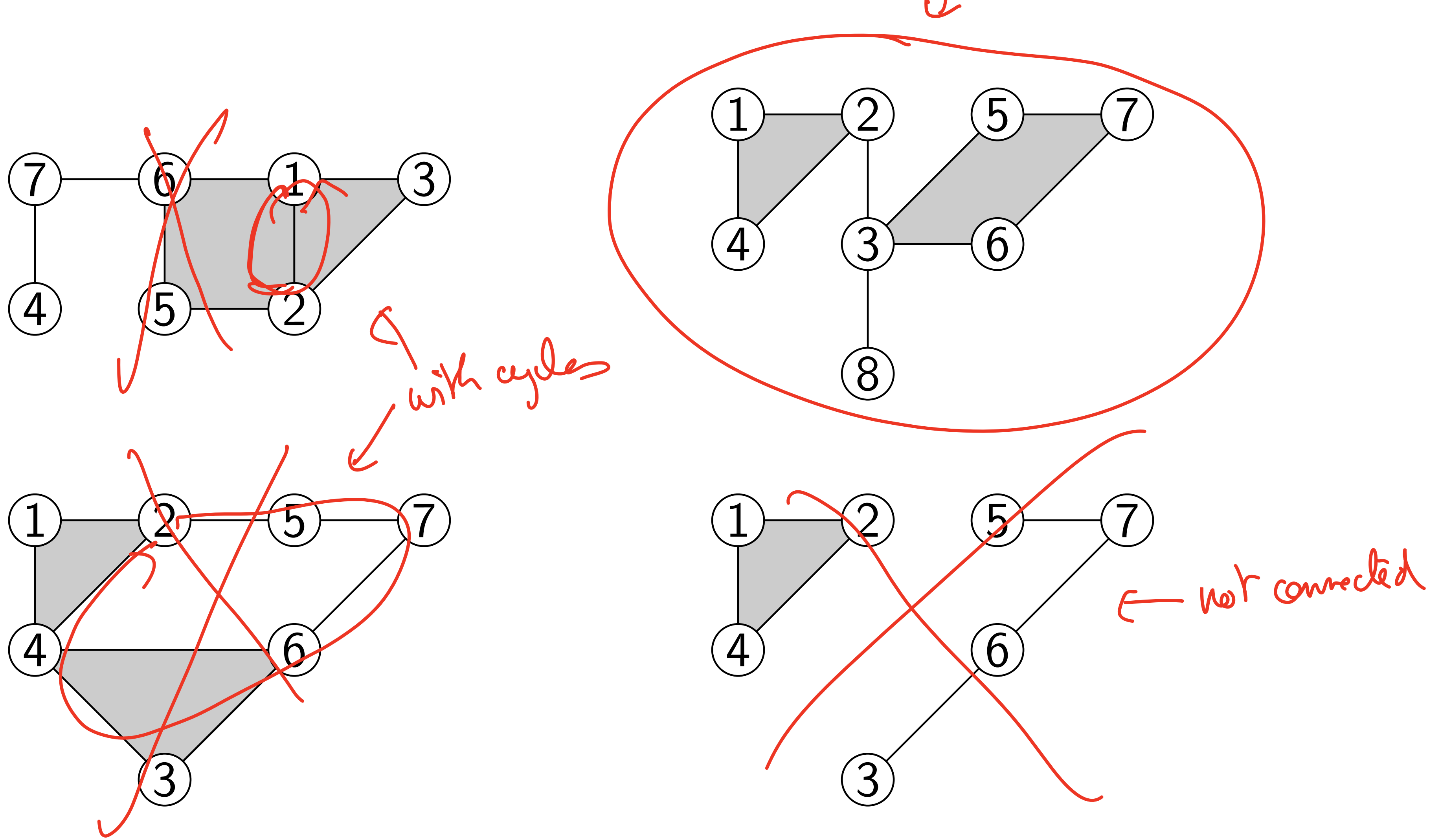
A hypertree is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

- there exists a walk from v to w in H with distinct edges e_i, (H is connected),
- and this walk is unique, (H has no cycles).

Example of a hypertree



First exercice : Which one(s) is/are a hypertree(s) ?



Numerology [Kalikow, 1999 ; Smith–Warme, 1998]

The number of hypertrees on n vertices is given by:

$$|HT_n| = \sum_{k=1}^{n-1} n^{k-1} S(n-1,k)$$

we consider that hypertrees are rooted in 1

partitions of {2,...,n} in k parts

S(n-1,n)=0

← Stirling number of the second kind = nb of partition of {1,...,n-1} in k parts

word of size k-1 over the alph {1,...,n}

A030019 Number of labeled spanning trees in the complete hypergraph on n vertices (all hyperedges having cardinality 2 or greater). 86

1, 1, 1, 4, 29, 311, 4447, 79745, 1722681, 43578820, 1264185051, 41381702275, 1509114454597, 60681141052273, 2667370764248023, 127258109992533616, 6549338612837162225, 361680134713529977507, 21333858798449021030515, 1338681172839439064846881 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,4

COMMENTS Equivalently, this is the number of "hypertrees" on n labeled nodes, i.e. connected hypergraphs that have no cycles, assuming that each edge contains at least two vertices. - Don Knuth, Jan 26 2008. See A134954 for hyperforests.

Also number of labeled connected graphs where every block is a complete graph (cf. A035053).

Let H = (V,E) be the complete hypergraph on N labeled vertices (all edges having cardinality 2 or greater). Let e in E and K = |e|. Then the number of distinct spanning trees of H that contain edge e is g(N,K) = K * E[X_N^{N-K}] / N and the K=1 case gives this sequence. Clearly there is some deep structural connection between spanning trees in hypergraphs and Poisson moments.

REFERENCES Warren D. Smith and David Warme, Paper in preparation, 2002.

LINKS Alois P. Heinz, Table of n, a(n) for n = 0..370 (first 101 terms from T. D. Noe)

Ayomikun Adeniran and Catherine Yan, Gončarov Polynomials in Partition Lattices and Exponential Families, arXiv:1907.07814 [math.CO], 2019.

Ronald Bacher, On the enumeration of labelled hypertrees and of labelled bipartite trees, arXiv:1102.2708v1 [math.CO], 2011.

Maryam Bahrani and Jérémie Lumbroso, Enumerations, Forbidden Subgraph Characterizations, and the Split-Decomposition, arXiv:1608.01465 [math.CO]

Proof: Prüfer code [R. Bacher 2014]

A **rooted hypertree** is a hypertree with a distinguished vertex.

↳ Here hypertrees are considered to be rooted in 1.

The **petiole** of an edge e is the closest vertex to the root in e.

= on any walk from a vertex of e to the root

We order edges according to $\min(e - \text{petiole}(e))$.

- Hypertree → Prüfer code: Let H be a hypertree with k edges on $\{1,\dots,n\}$ rooted in 1

  Set of edges \ petiole = partition of $\{2,\dots,n\}$ in k parts

  How to remember the shape of the hypertree? → Prüfer code

Algorithm: $w \leftarrow \varepsilon$ (Initialization of the word)

$\tilde{H} \leftarrow H$

While $\tilde{H}$ has strictly more than 1 edge:

- $e \leftarrow$ smallest leaf of $\tilde{H}$
- $w \leftarrow w + \text{petiole}(e)$
- $\tilde{H} \leftarrow \tilde{H} \setminus (e \setminus \text{petiole}(e))$

  = we remove the vertices of $e \setminus \text{petiole}(e)$ from $\tilde{H}$

return w ← Prüfer code of the hypertree

Exple:

$\pi = \{2\}\{39\}\{46\}\{5\}\{7\}\{8\}$

⟷

$w = 98441$

- Prüfer code (+ set partition) → hypertree

Start with a partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of $\{2, \ldots, n\}$ and a word $w_1 \ldots w_k$ on $\{1, \ldots, n\}$.

We construct iteratively a hypertree.
More precisely at each iteration, we add an edge to the hypertree we are constructing.

Algorithm: $\tilde{\pi} \leftarrow \pi$ (will contain the parts of the partition which have to be linked to a vertex to form an edge in the hypertree)

For i from 1 to $k-1$:
  $p \leftarrow \min\{\pi_j \in \tilde{\pi}$ s.t. no letter of $w_i \ldots w_k$ is in $\pi_j\}$
  we add the edge $(p \cup w_i)$ to the hypertree we are constructing
  and remove p from $\tilde{\pi}$

Add the edge $\pi_1 \cup \{1\}$ for the only π_1 remaining in $\tilde{\pi}$

- The condition "no letter in $w_i \ldots w_k$ is in π_j" ensures the acyclicity of the obtained hypergraph.

Moreover, every vertex belongs to one of the added edges: we can show by induction on k that the obtained hypergraph is connected and hence that we get a hypertree

The hypertree poset

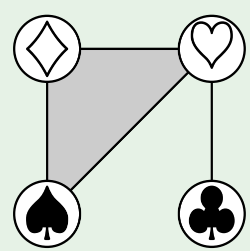
Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

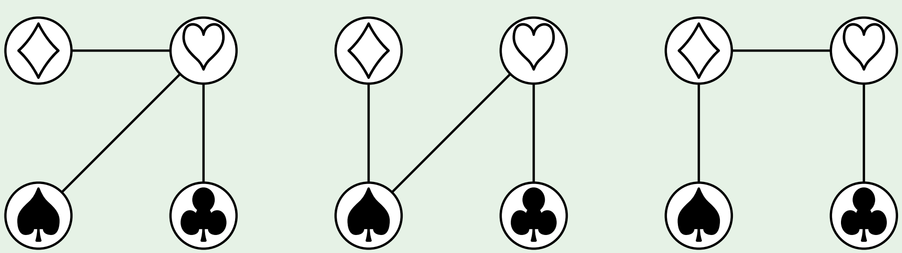
$$S \leq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \leq T$ but $S \neq T$.

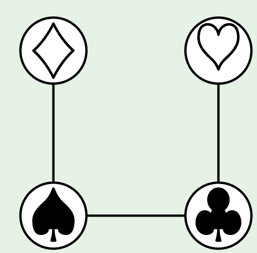
Example with hypertrees on four vertices



$\preceq$



but not



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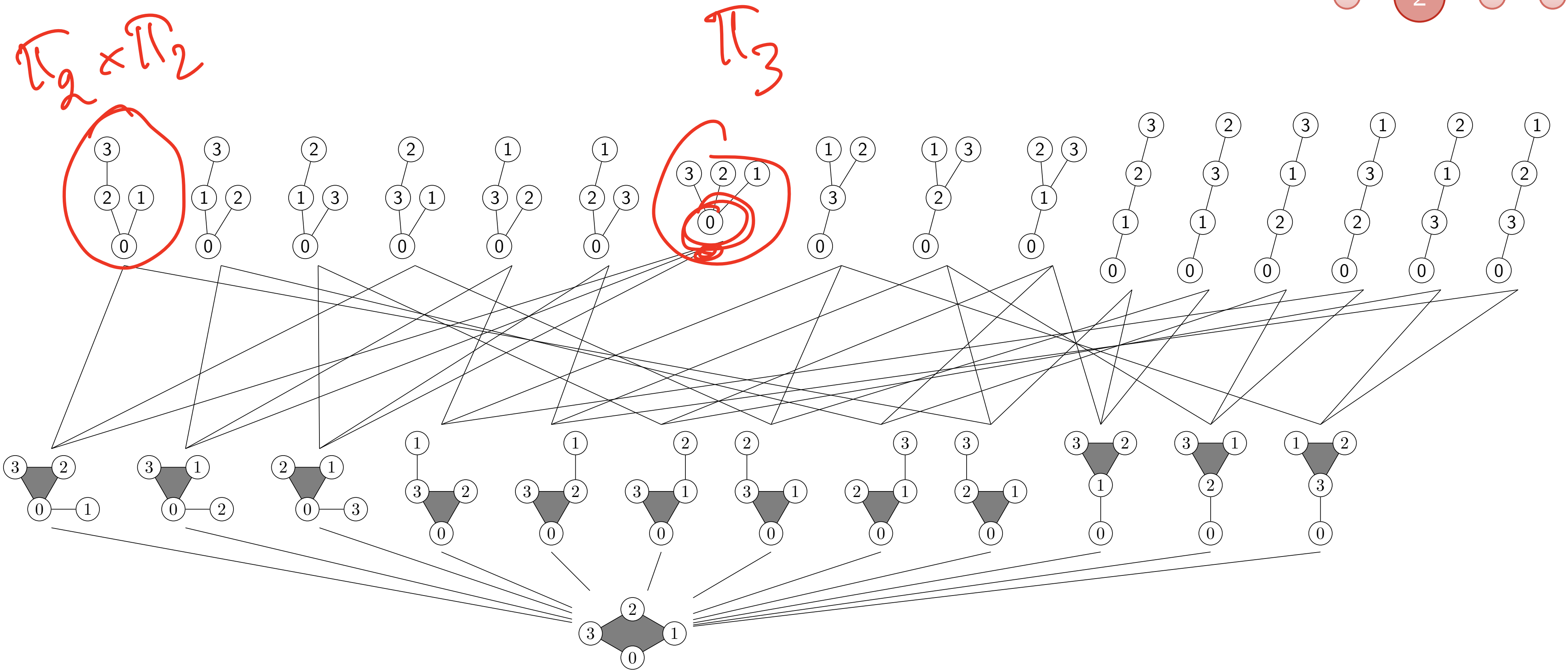

- $\widehat{\mathsf{HT}}_{\mathsf{n}}$ = augmented hypertree poset on $[\mathbf{0}, n]$.
- HT_{n} = hypertree poset on $[\mathbf{0}, n]$.
- For a a tree in HT_{n}, $\mathsf{HT}^{\mathsf{a}}_{\mathsf{n}}$ = maximal interval in hypertree poset on $[\mathbf{0}, n]$ between $\hat{0}$ and a.

Incidence Hopf algebra of the hypertree posets

Main question

What is the shape of an interval in HT_n ?

Proposition (McCammond–Meier, 2004)

Let H be a hypertree on n vertices and a be a tree such that $H \leqslant a$. The following isomorphisms hold:

$$[0_{\mathrm{HT}_n}, H] \simeq \prod_{v \in V(H)} \Pi_{\deg(v)} \qquad [H, a] \simeq \prod_{e \in E(H)} \mathrm{HT}_e^{a_{|e}}.$$

In particular, $\mathrm{HT}_n^a = \prod_{v \in V(a)} \Pi_{\deg(v)}$.

Möbius number of the hypertree posets

Exercise prove this (with $x^n = \sum_{k=0}^{n} (-1)^{n-k} S(n,k)\, x(x+1)\dots(x+k-1)$)

Proposition (McCammond-Meier, 2004)

The Möbius number of $\widehat{\mathrm{HT}}_n$ *is given by:*

$$\mu(\widehat{\mathrm{HT}}_n) = (-1)^{n-1}(n-1)^{n-2}$$

$(n-1)^{n-2}$ = nb of Cayley trees on $(n-1)$ vertices = dim PreLie$(n-1)$

Proposition

The Möbius number of HT_n *is given by:*

$$\mu(\mathrm{HT}_n) = (-1)^n \frac{(2n-3)!}{(n-1)!}$$

$\frac{(2n-3)!}{(n-1)!}$ = dim PostLie$(n-1)$

$\frac{(2n-3)!}{(n-1)!}$?

A006963 Number of planar embedded labeled trees with n nodes: (2n-3)!/(n-1)! for n >= 2, a(1) = 1. 28

(Formerly M3076)

1, 1, 3, 20, 210, 3024, 55440, 1235520, 32432400, 980179200, 33522128640, 1279935820800, 53970627110400, 2490952020480000, 124903451312640000, 6761440164390912000, 393008709555221760000, 24412776311194951680000, 1613955767240110694400000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS For n>1: central terms of the triangle in A173333; cf. A001761, A001813. - Reinhard Zumkeller, Feb 19 2010

Can be obtained from the Vandermonde permanent of the first n positive integers; see A093883. - Clark Kimberling, Jan 02 2012

All trees can be embedded in the plane, but "planar embedded" means that orientation matters but rotation doesn't. For example, the n-star with n-1 edges has n! ways to label it, but rotation removes a factor of n-1. Another example, the n-path has n! ways to label it, but rotation removes a factor of 2. - Michael Somos, Aug 19 2014

REFERENCES N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

LINKS Vincenzo Librandi, Table of n, a(n) for n = 1..200

David Callan, A quick count of plane (or planar embedded) labeled trees.

Ali Chouria, Vlad-Florin Drăgoi, and Jean-Gabriel Luque, On recursively defined combinatorial classes and labelled trees, arXiv:2004.04203 [math.CO], 2020.

Robert Coquereaux and Jean-Bernard Zuber, Maps, immersions and permutations, Journal of Knot Theory and Its Ramifications, Vol. 25, No. 8 (2016), 1650047; arXiv preprint, arXiv:1507.03163 [math.CO], 2015-2016.

INRIA Algorithms Project, Encyclopedia of Combinatorial Structures 109.

Bradley Robert Jones, On tree hook length formulas, Feynman rules and B-series, Master's thesis, Simon Fraser University, 2014.

Pierre Leroux and Brahim Miloudi, Généralisations de la formule d'Otter, Ann. Sci.

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Pierre Leroux and Brahim Miloudi, Généralisations de la formule d'Otter, Ann. Sci. Math. Québec, Vol. 16, No. 1 (1992), pp. 53-80. (Annotated scanned copy)

J. W. Moon, Counting Labelled Trees, Issue 1 of Canadian mathematical monographs, Canadian Mathematical Congress, 1970.

Ran J. Tessler, A Cayley-type identity for trees, arXiv:1809.00001 [math.CO], 2018.

Index entries for sequences related to trees.

FORMULA

E.g.f. for a(n+1), n >= 1, log(c(x)); c(x) = g.f. for Catalan numbers A000108. - Wolfdieter Lang

Integral representation as n-th moment of a positive function on a positive half-

Lie ∘ Mag

Prop: $\mu(\mathcal{H}T_n) = (-1)^n \frac{(2n-3)!}{(n-1)!}$

Proof: $\mu(\mathcal{H}T_n) = \sum_{a \text{ tree of size } n} \mu(\mathcal{H}T_n^a)$

$$= \sum_{a \text{ tree of size } n} \prod_{v \in V(a)} \mu(\Pi_{deg(v)})$$

$$= \sum_{d_1 + \dots + d_n = n-2} \frac{(n-2)!}{d_1! \dots d_n!} \prod_{i=1}^{n} (-1)^{d_i} d_i!$$

$$= (-1)^{n-2} (n-2)! \underbrace{\sum_{d_1 + \dots + d_n = n-2} 1}_{= \binom{2n-3}{n-2}}$$

$$= (-1)^n \frac{(2n-3)!}{(n-1)!}$$

Prop: $\mu(\widehat{\mathcal{H}T_n}) = (-1)^{n-1} (n-1)^{n-2}$

Proof: $\mu(\widehat{\mathcal{H}T_n}) = -\sum_{H \in \mathcal{H}T_n} \mu([\hat{0}_{\mathcal{H}T_n}, H])$

$$= -\sum_{H \in \mathcal{H}T_n} \prod_{v \in V(H)} \mu(\Pi_{deg(v)})$$

$$= -\sum_{k=1}^{n-1} S(n-1,k) \times \sum_{d_1 + \dots + d_n = k-1} \frac{(k-1)!}{d_1! \dots d_n!} \prod_{i=1}^{n} (-1)^{d_i} d_i!$$

(d_1: nb of 1 in the Prüfer code; d_n: nbr of n in the Prüfer code)

$$= \sum_{k=1}^{n-1} S(n-1,k) (-1)^k (k-1)! \underbrace{\sum_{d_1 + \dots + d_n = k-1} 1}_{= \binom{n+k-2}{k-1}}$$

$$= \sum_{k=1}^{n-1} S(n-1,k)\,(-1)^k \underbrace{(n+k-2)\dots(n)}$$

$$= \frac{(n+k-2)!}{(n-1)!}$$

as $\quad x^\alpha = \sum_{k=0}^{\alpha} (-1)^{\alpha-k}\, S(\alpha,k)\,(x+k-1)\dots x$

taking $\begin{cases} \alpha = n-1 \\ x = n-1 \end{cases}$

we get: $\mu\,(\widehat{1, \Pi_n}) = (-1)^{n-1}\,\frac{(n-1)^{n-1}}{(n-1)} = (-1)^{n-1}(n-1)^{n-2}$

Reference(s)

- The hypertree poset and the l^2-Betti number of the motion group of the trivial link, J. McCammond and J. Meier, *Matematische Annalen, N°328, 633-652* (2004)

Operads and homology

Outline

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Outline