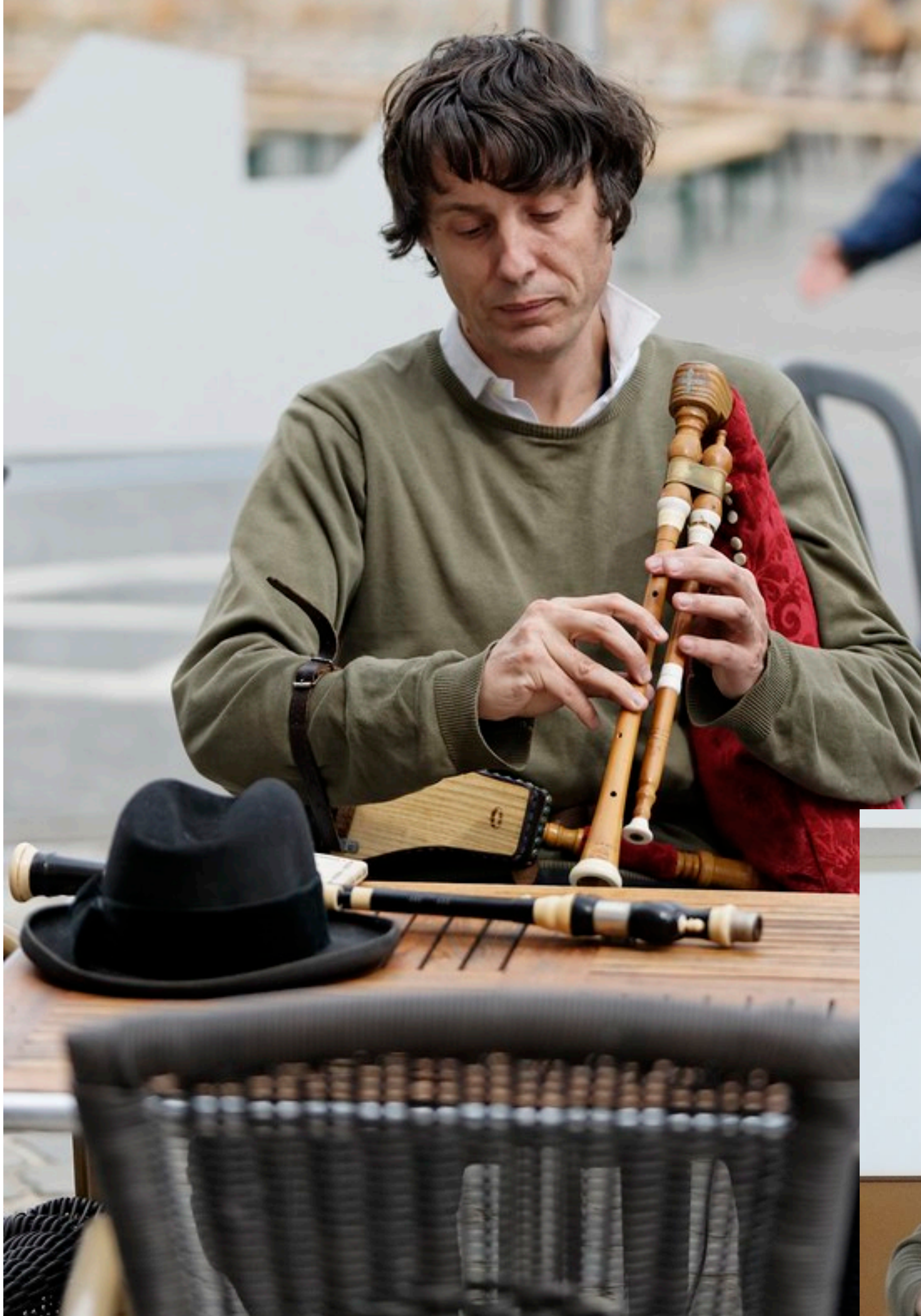
Posets, incidence Hopf algebras and operads

Bérénice Delcroix-Oger

Journées du GDR Renorm
Du 14 au 18 novembre 2022, Calais


CIRM
Free post-Lie algebras, the Hopf algebra
of Lie group integrators and planar arborification
Dominique Manchon



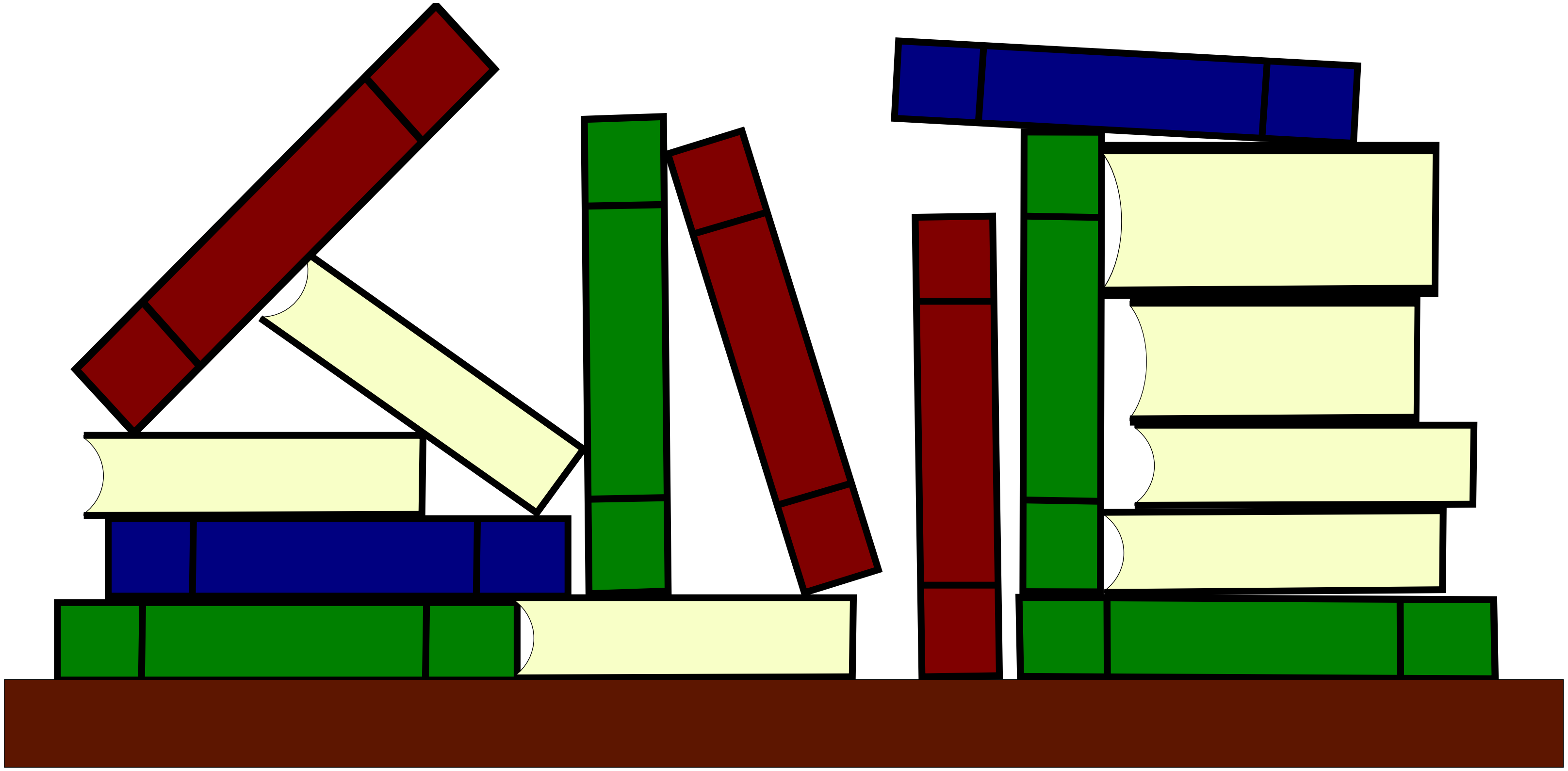
Outline

Posets and incidence Hopf algebra

Outline

1. Posets and incidence Hopf algebra
2. Hypertrees
3. Operads and homology
4. Back to the homology of the hypertree posets

Poset = Partially ordered set

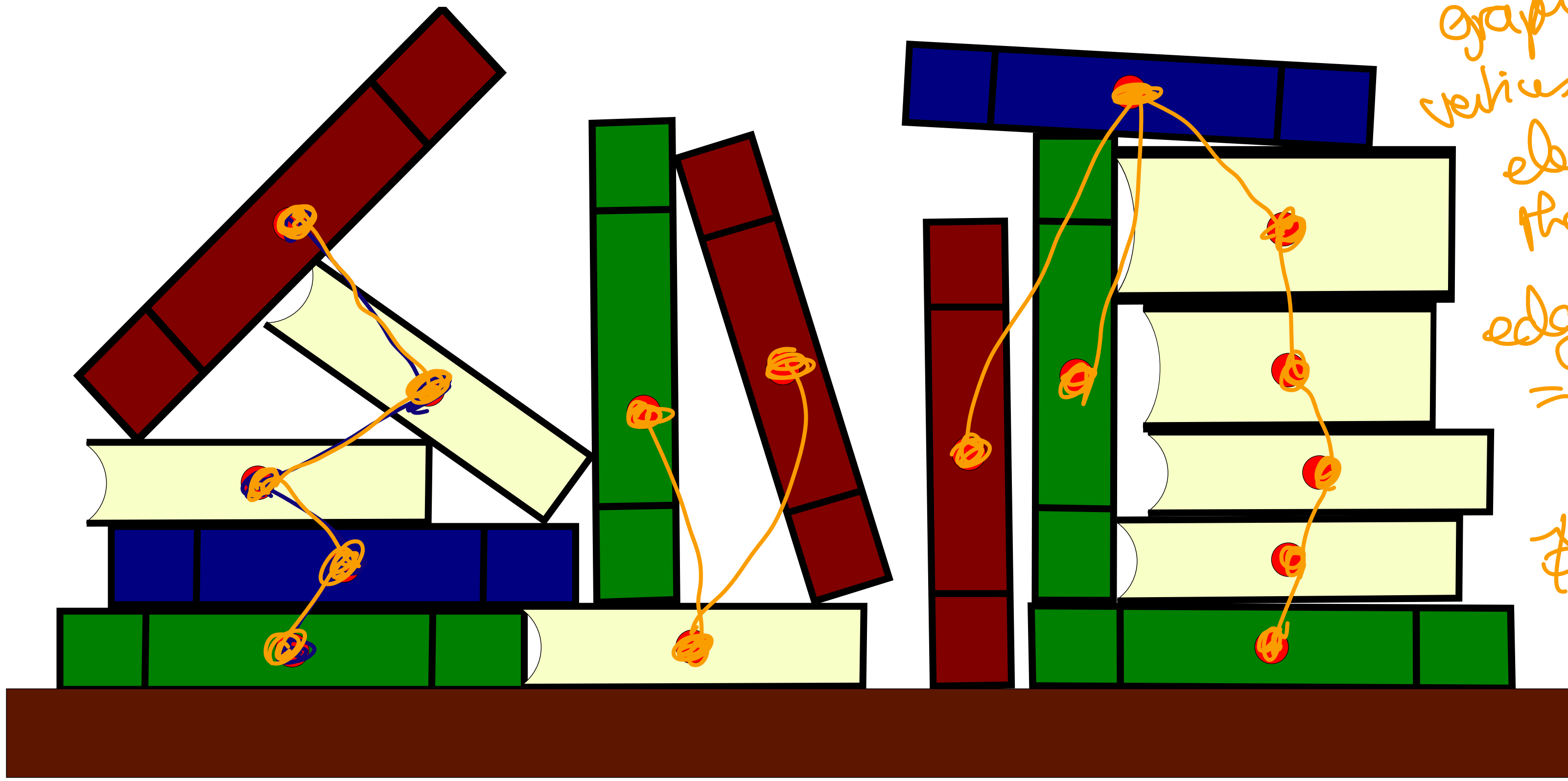


Poset = Partially ordered set (and their Hasse diagram)

$a \leq b$ if I have to remove b in order to remove a

graph
vertices = elements of the poset

edges = covering relations (a,b) $\nexists c : a < c < b$



Exercise 1

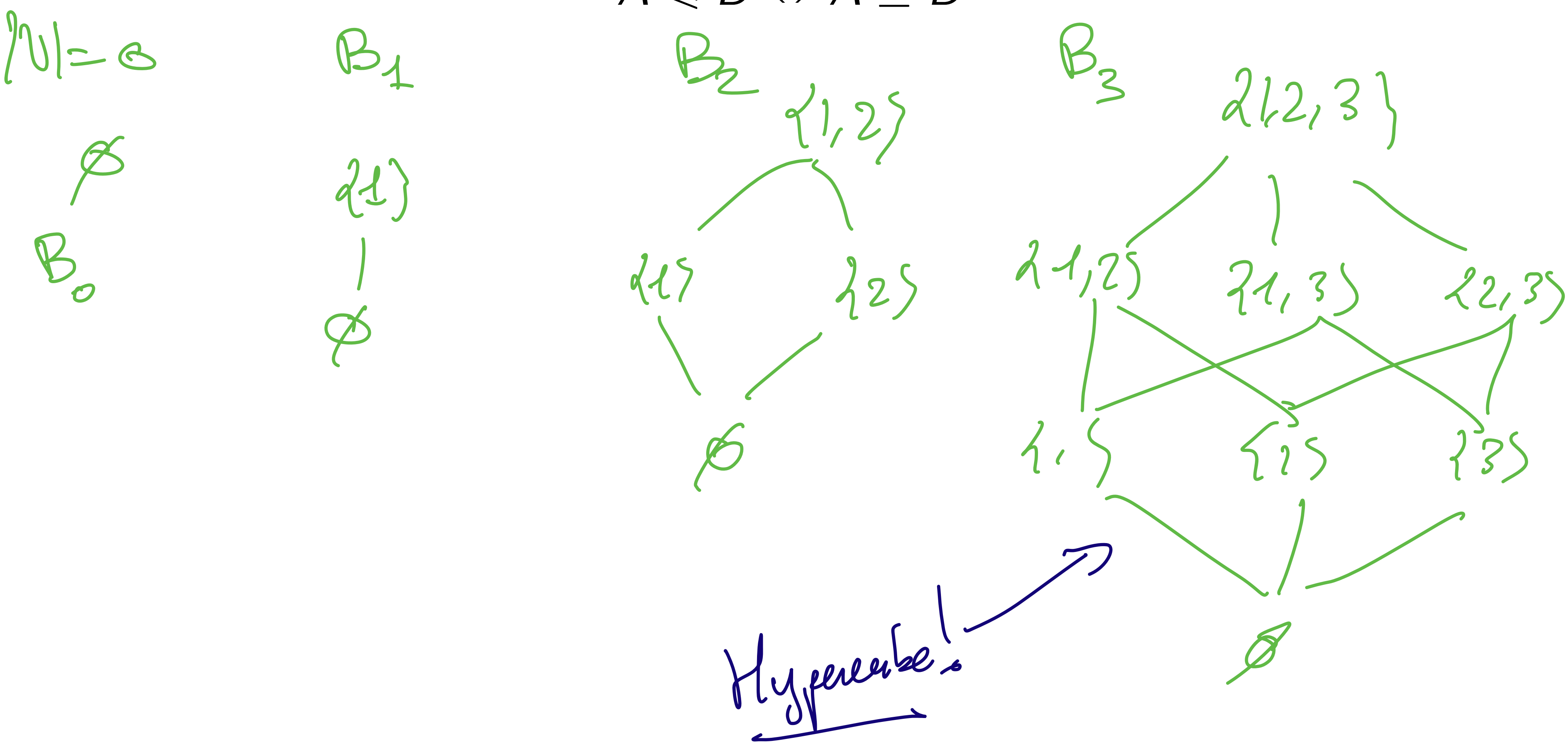
Draw the Hasse diagram of your favorite poset.

First poset: Boolean poset (or lattice)

Consider the set of subsets of a set V, with the partial order given by inclusion of subsets:

$\{1,\dots,n\}$

$$A \leqslant B \Leftrightarrow A \subseteq B$$



Posets of (set) partitions Π_V

$\Pi_1 = \{1\}$ $\Pi_2 =$ $\{1\}\{2\}$ | $\{1,2\}$

Partitions of a set V :

$$\{V_1, \ldots, V_k\} \models V \Leftrightarrow V = \bigsqcup_{i=1}^{k} V_i,\, V_i \cap V_j = \varnothing \text{ for } i \neq j$$

Partial order on set partitions of a set V:

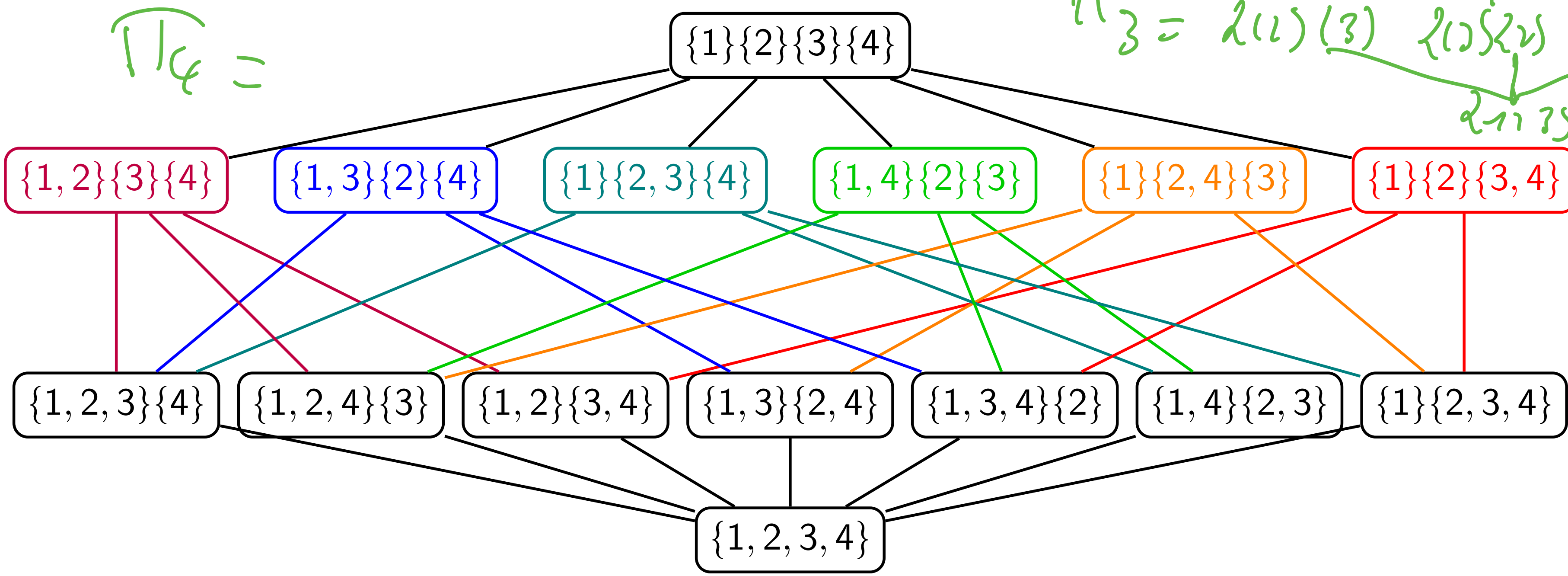
$$\{V_1, \ldots, V_k\} \leqslant \{V'_1, \ldots, V'_p\} \Leftrightarrow \forall i \in \{1, \ldots, p\}, \exists j \in \{1, \ldots, k\} \text{ s.t. } V'_i \subseteq V_j$$

$\{1\}\{2\}\{3\}$

$\Pi_3 =$ $\{1,2\}\{3\}$ $\{1,3\}\{2\}$ $\{1\}\{2,3\}$

$\{1,2,3\}$

$\Pi_4 =$

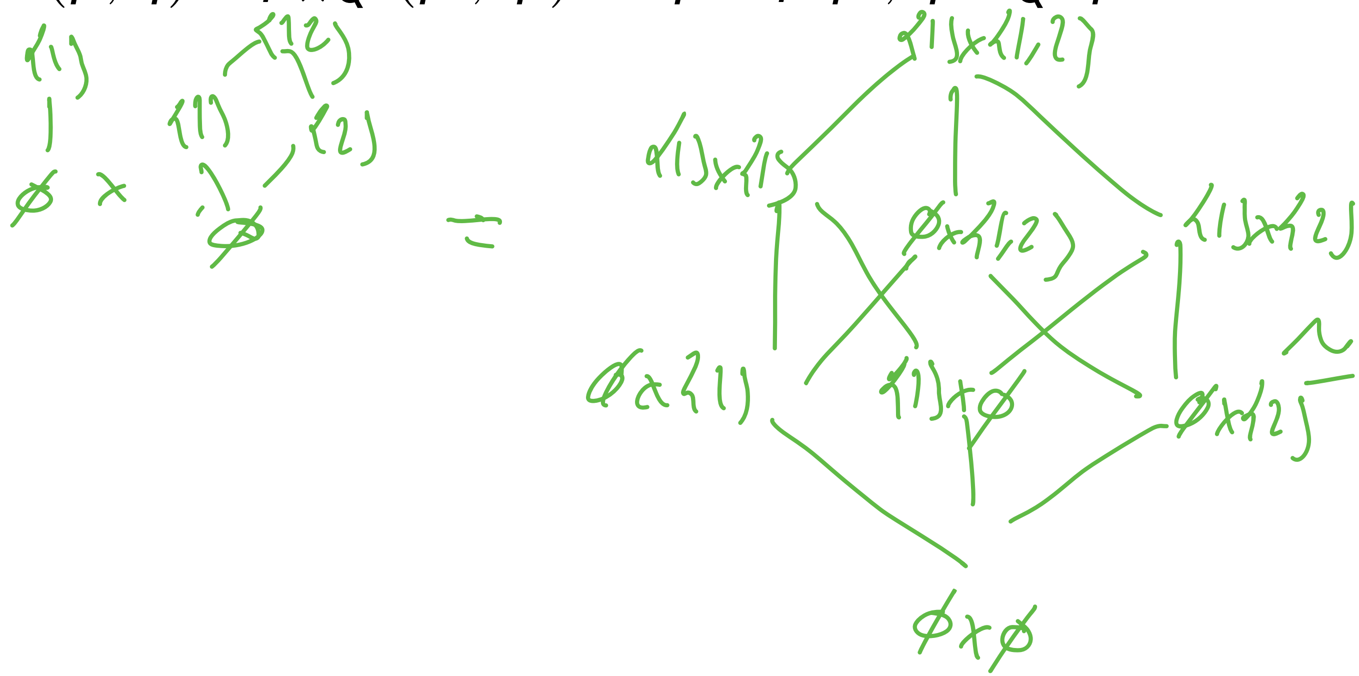


Operations on posets : Cartesian product

If P and Q are two posets, their cartesian product is the set $P \times Q$ endowed with the following partial order:

$$(p, q) \leqslant_{P \times Q} (p', q') \Leftrightarrow p \leqslant_P p', q \leqslant_Q q'$$

$B_1 \times B_2 =$



$\simeq B_3$

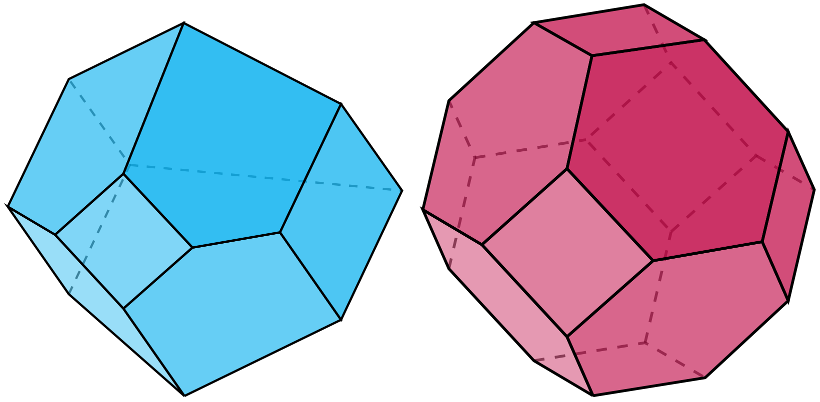
Operations on posets : Isomorphisms of posets

''
relabeling

Two posets P and Q are isomorphic if there exists an order-preserving bijection $f : P \to Q$, i.e. a bijection f such that $f(a) \leqslant_Q f(b)$ iff $a \leqslant_P b$.

Why do we look at posets ?

- They appear in various places.



- They are linked with some topological/algebraic invariants.
- They are FUN !!!

Hopf incidence algebra of a (finite) poset [Rota 1964, Schmitt 1992]

Let us consider a family of finite bounded posets $\mathcal{F}_P$

closed under subintervals $(\forall p \in \mathcal{F}_P, \forall x \leqslant y \in p, [x; y] \in \mathcal{F}_P)$

closed by product $(\forall p, q \in \mathcal{F}_P, p \times q \in \mathcal{F}_P)$

Examples

The family of boolean posets and the family of cartesian products of partition posets satisfy these properties.

Coproduct of the algebra

Given $\mathbb{C}$ your favorite commutative ring with unit (for instance $\mathbb{C}$), define

$$\mathcal{C} := \mathbb{C}.\mathcal{F}_P / \sim,$$

the free $\mathbb{C}$-vector space on the quotient $\mathcal{F}_P$ by isomorphism classes of posets.
$\mathcal{C}$ is endowed with the coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and the counit $\epsilon : \mathcal{C} \to \mathbb{C}$ defined by:

$$\Delta(P) = \sum_{x \in P} [0_P; x] \otimes [x, 1_P]$$

$$\epsilon(P) = \delta_{|P|=1}$$

Theorem (Schmitt)

$(\mathcal{C}, \Delta, \epsilon, \times, \nu, S)$ *is a Hopf algebra.*

Incidence Hopf algebra of the boolean lattice

Let $V \in B_n$, $V = \{i_1, \ldots, i_k\}$

Lemma

The following isomorphisms hold:

$$[V, \{1, \ldots, n\}] \simeq B_{n-k} \qquad [\varnothing, V] \simeq B_k$$

The coproduct is given by:

$$\Delta(B_n) = \sum_{k=0}^{n} \binom{n}{k} B_k \otimes B_{n-k}.$$

Incidence Hopf algebra of the poset of partitions

Let $\pi \in \Pi_n$, $\pi = \{V_1, \ldots, V_k\}$

Lemma

The following isomorphisms hold:

$$[\pi, 1_{\Pi_n}] \simeq \prod_{i=1}^{k} \Pi_{|V_k|} \qquad [0_{\Pi_n}, \pi] \simeq \Pi_k$$

$$\Delta\left(\frac{\Pi_n}{n!}\right) = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_n)\in\mathbb{N}, \sum_{i=1}^{n} j_i = k, \sum_{i=1}^{n} i j_i = n} \binom{k}{j_1, \ldots, j_n} \prod_{i=1}^{n} \left(\frac{\Pi_i}{i!}\right)^{j_i} \otimes \frac{\Pi_k}{k!}.$$

But...

Does this formula seem familiar to you ?

Some parenthesis : Faà di Bruno Hopf algebra [Joni-Rota 1982]

Consider $\mathcal{E}$, the ring of exponential formal series $f(t) = \sum_{n=1}^{\infty} \frac{f_n}{n!} t^n$, with $f_1 > 0$ (endowed with the substitution product). Let us define the characters $a_n(f) := f_n$, for $n \geqslant 1$.

What is the value of $a_n(f \circ g)$ in terms of $a_n(f)$ and $a_n(g)$?

$$\frac{a_n(f \circ g)}{n!} = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_n)\in\mathbb{N}, \sum_{i=1}^{n} j_i = k, \sum_{i=1}^{n} i j_i = n} \binom{k}{j_1, \ldots, j_n} \prod_{i=1}^{n} \left(\frac{a_i(g)}{i!}\right)^{j_i} \frac{a_k(f)}{k!}$$

Defining $\Delta a_n(g, f) = a_n(f \circ g)$, we get the "familiar" formula !

Your second exercise

What about the incidence Hopf algebra of the boolean posets ?

$$\Delta\left(B_n\right) = \sum_{k=0}^{n} \binom{n}{k} B_k \otimes B_{n-k}.$$

Character of an incidence Hopf algebra

Consider the vector space of characters $\mathcal{H}^* = \mathrm{Hom}(\mathcal{H}, \mathbb{C})$ on an incidence Hopf algebra $\mathcal{H}$.
The convolution of two characters ϕ and ψ is given by:

$$\phi * \psi = \sum \phi(P_{(1)})\psi(P_{(2)})$$

where $\Delta(P) = \sum P_{(1)} \otimes P_{(2)}$.

On the boolean lattice

The vector space of characters on the incidence Hopf algebra of the partition posets corresponds to (No, I won't give here the answer to the exercise !)

On the partition lattice

The vector space of characters on the incidence Hopf algebra of the partition posets corresponds to exponential generating functions via $\phi \mapsto \sum_{n\geqslant 1} \frac{\phi(\Pi_n)}{n!} t^n$.

Some basic characters

Let us consider the character

$$\xi : \Pi_n \mapsto 1.$$

and let μ be its inverse for the convolution product.

For partitions

We have $\xi(t) = \sum_{n\geqslant 1} \frac{\xi(\Pi_n)}{n!} t^n = \sum_{n\geqslant 1} \frac{1}{n!} t^n = \exp(t) - 1$ and $\mu(t) = \ln(1+t) = \sum_{n\geqslant 1} (-1)^{n-1}(n-1)! \frac{t^n}{n!}$

Reference(s)

- Incidence Hopf algebras, W. Schmitt, *J. Pure Appl. Algebra 96, N°3, 299-330* (1994)

END

of 1st lecture

Hypertrees

Outline

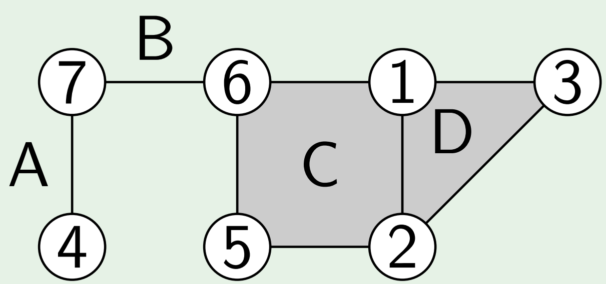
Hypergraphs

Definition (Berge)

A hypergraph (on a set V) is an ordered pair (V, E) where:

- V is a finite set (vertices)
- E is a collection of subsets of cardinality at least two of elements of V (edges).

Example of a hypergraph on $[1; 7]$



Walk on a hypergraph

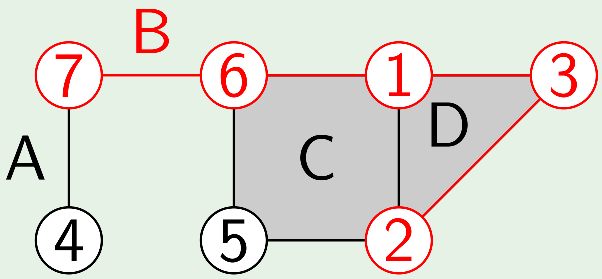
Definition

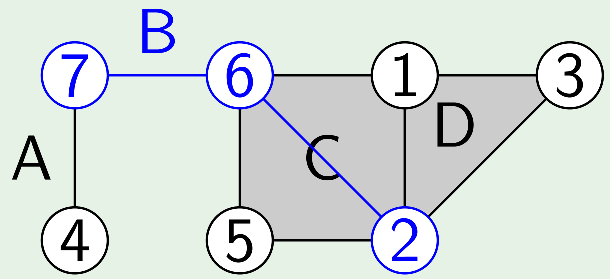
Let $H = (V, E)$ be a hypergraph.
A walk from a vertex or an edge d to a vertex or an edge f in H is an alternating sequence of vertices and edges beginning by d and ending by f:

$$(d, \ldots, e_i, v_i, e_{i+1}, \ldots, f)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.
The length of a walk is the number of edges and vertices in the walk.

Examples of walks





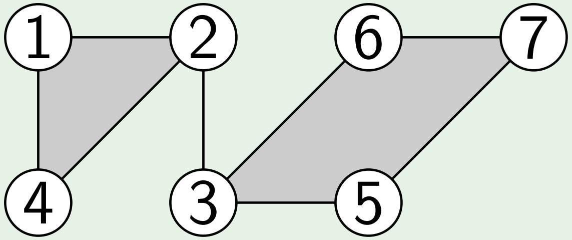
Hypertrees

Definition

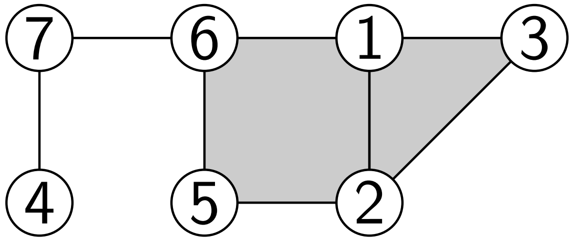
A hypertree is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

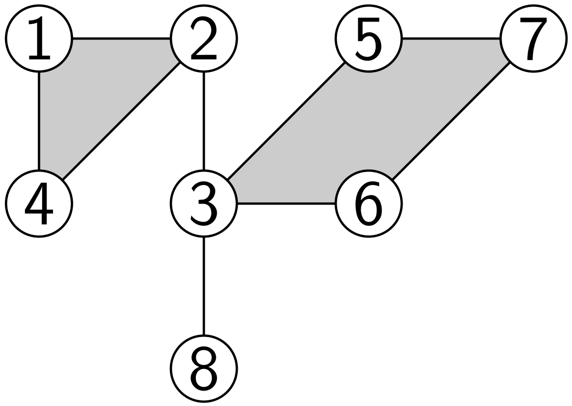
- there exists a walk from v to w in H with distinct edges e_i, (H is connected),
- and this walk is unique, (H has no cycles).

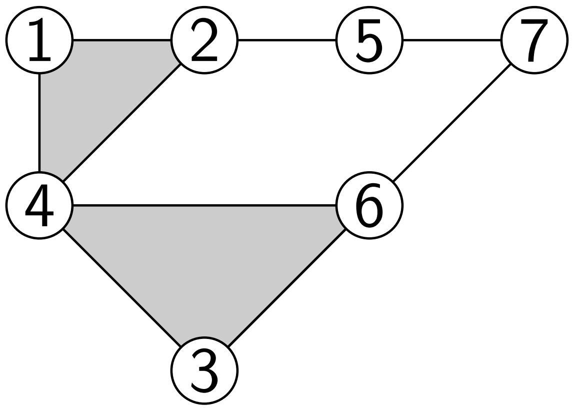
Example of a hypertree

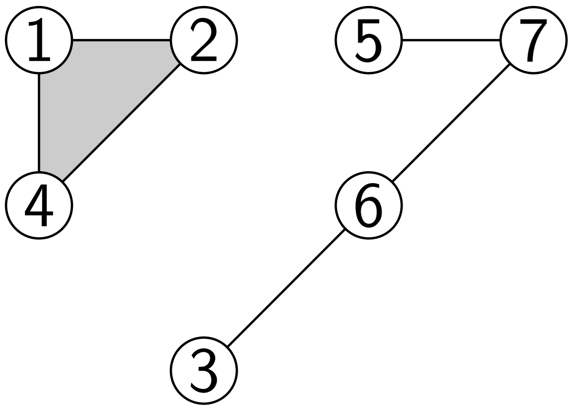


First exercice : Which one is a hypertree ?









The hypertree poset

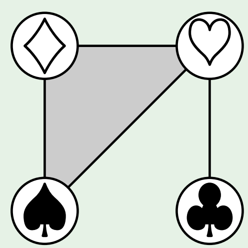
Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

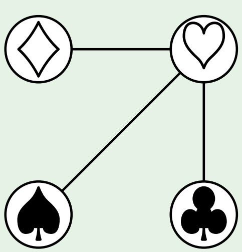
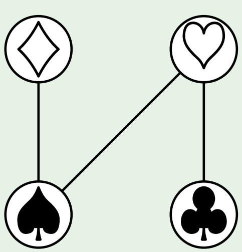
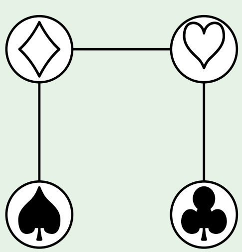
$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

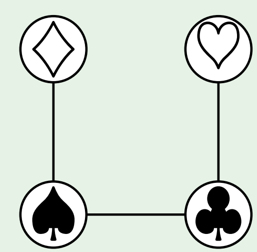
Example with hypertrees on four vertices



$\preceq$

but not



.

Operads and homology

Outline

Back to the homology of the hypertree posets

Outline