A spoonful of dendrology: from hypertrees to Cayley trees

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Algebraic Combinatorics of the Symmetric Groups and Coxeter Groups II
Cetraro, July 2022

Outline

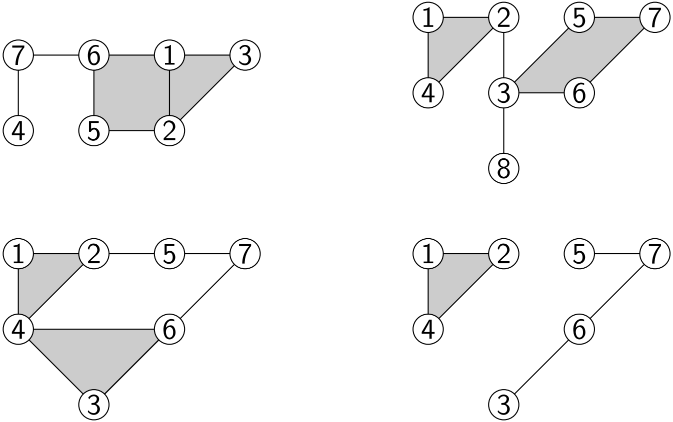
1. From hypertrees to n^{n-1}
2. Action of the symmetric group on this poset

From hypertrees to n^{n-1}

Outline

1. From hypertrees to n^{n-1}
   - Hypertree poset
   - Homology of the hypertree posets
   - From hypertrees to Cayley trees
2. Action of the symmetric group on this poset

Hypergraphs [Berge, 80s] and hypertrees



The hypertree poset

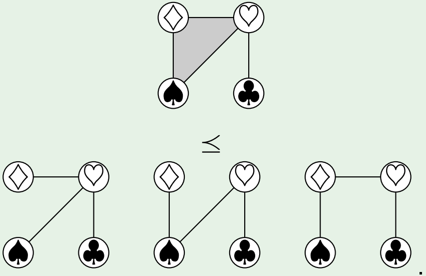
Definition

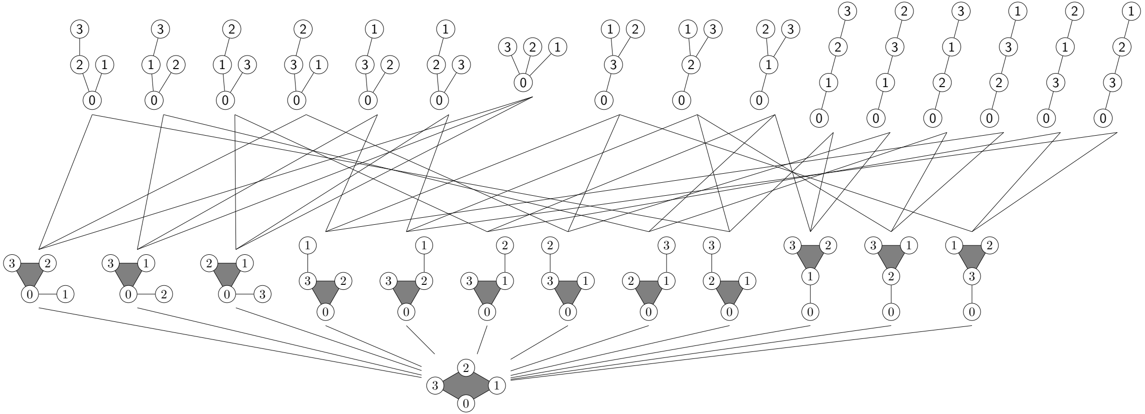
Let I be a finite set of cardinality n, S and T be two hypertrees on I.

$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

Example with hypertrees on four vertices





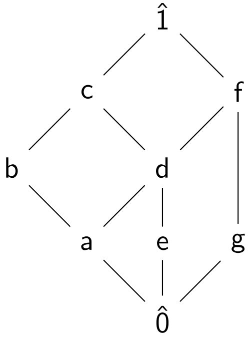
- HT_n = hypertree poset on $[\mathbf{0}, n]$.
- $\widehat{HT_n}$ = augmented hypertree poset on $[\mathbf{0}, n]$.

Goal of this section:

- Compute of the action of the symmetric group on the homology of this poset

Homology of the poset

To each poset, we can associate a simplicial complex (order complex).



The homology of the poset is the (reduced) homology of this simplicial complex (topological invariant).

Theorem (McCammond-Meier, 04)

The homology of $\widehat{\mathrm{HT}_\mathrm{n}}$ *and* HT_n *are concentrated in maximal degree* $n-2$ *and* $n-3$ *respectively.*

Corollary

The character for the action of the symmetric group on $\tilde{H}_{n-3}$ *is given in terms of characters for the action of the symmetric group on* C_k *by:*

$$\chi_{\tilde{H}_{n-3}} = (-1)^{n-3} \sum_{k=-1}^{n-3} (-1)^k \chi_{C_k}, \text{ where } n = \#I,$$

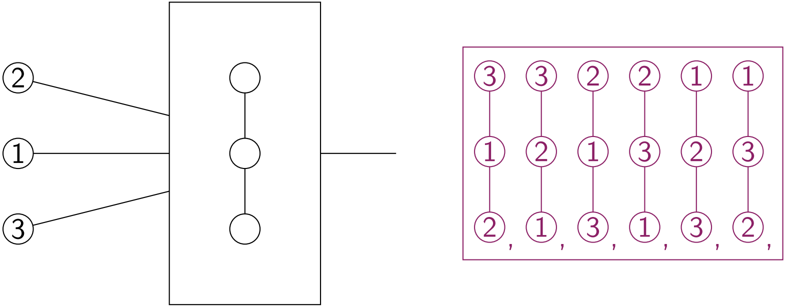
where C_k *is the vector space spanned by* $k+1$*-tuples* $(a_0, \ldots, a_k)$ *with* $a_i < a_{i+1}$.

What are species?

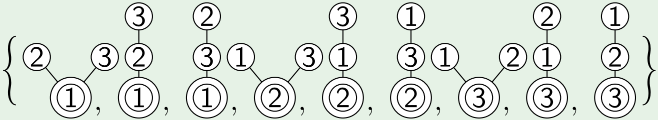
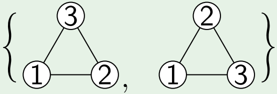
Definition (Joyal, 80s)

A species F is a functor from the category of finite sets and bijections to itself. To a finite set I, the species F associates a finite set $F(I)$ independent from the nature of I.

Species = Construction plan, such that the set obtained is invariant by relabelling

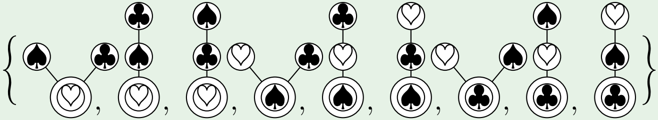
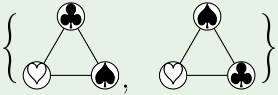


Examples of species

- $\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists $\mathbb{L}$ on $\{1,2,3\}$)
- $\{\{1,2,3\}\}$ (species of non-empty sets $\mathbb{E}^+$)
- $\{\{1\},\{2\},\{3\}\}$ (species of pointed sets $\mathbb{E}^\bullet$)
-  (Species of Cayley trees $\mathbb{C}$)
-  (Species of cycles)

These sets are the image by species of the set $\{1,2,3\}$.

Examples of species

- $\{(\heartsuit,\spadesuit,\clubsuit),(\heartsuit,\clubsuit,\spadesuit),(\spadesuit,\heartsuit,\clubsuit),(\spadesuit,\clubsuit,\heartsuit),(\clubsuit,\heartsuit,\spadesuit),(\clubsuit,\spadesuit,\heartsuit)\}$ (Species of lists $\mathbb{L}$ on $\{\clubsuit,\heartsuit,\spadesuit\}$)
- $\{\{\heartsuit,\spadesuit,\clubsuit\}\}$ (Species of non-empty sets $\mathbb{E}^+$)
- $\{\{\heartsuit\},\{\spadesuit\},\{\clubsuit\}\}$ (Species of pointed sets $\mathbb{E}^\bullet$)
-  (Species of rooted trees $\mathbb{C}$)
-  (Species of cycles)

These sets are the image by species of the set $\{\clubsuit,\heartsuit,\spadesuit\}$.

Operations on species and generating series

Proposition

Let F and G be two species. Let us define:

$$(F+G)(I) = F(I) \sqcup G(I),$$
$$(F \times G)(I) = \bigsqcup_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2),$$
$$(F \circ G)(I) = \bigsqcup_{\pi \in \mathcal{P}(I)} F(\pi) \times \prod_{J \in \pi} G(J),$$

where $\mathcal{P}(I)$ runs on the set of partitions of I.

Definition

To a species F, we associate its generating series:

$$C_F(x) = \sum_{n \geqslant 0} \#F(\{1, \ldots, n\}) \frac{x^n}{n!}.$$

Examples of generating series:

- The generating series of the species of lists is $C_{\mathbb{L}} = \frac{1}{1-x}$.
- The generating series of the species of non-empty sets is $C_{\mathbb{E}^+} = \exp(x) - 1$.
- The generating series of the species of pointed sets is $C_{\mathbb{E}^\bullet} = x \cdot \exp(x)$.
- The generating series of the species of rooted trees is $C_{\mathbb{C}} = \sum_{n \geqslant 0} n^{n-1} \frac{x^n}{n!}$.
- The generating series of the species of cycles is $C_{Cycles} = -\ln(1-x)$.

Counting strict chains using large chains

Let I be a finite set of cardinality n.

Definition

A large k-chain of hypertrees on I is a k-tuple $(a_1, \ldots, a_k)$, where a_i are hypertrees on I and $a_i \preceq a_{i+1}$.

We get, for all integer $k > 0$:

$$\chi_k = \sum_{i=0}^{n-2} \binom{k}{i} \chi_{C_i}.$$

χ_k is a polynomial $P(k)$ in k which gives, once evaluated in -1, the character:

Corollary

$$\chi_{\tilde{H}_{n-2}(\widehat{\mathrm{HT_n}})} = (-1)^n P(-1) =: (-1)^n \chi_{-1}$$

From hypertrees to Cayley trees

We denote by:

- $\mathcal{H}_k^0$, the species which associated to any finite set V the set of large k-chains of hypertrees on $\{0\} \cup V$,
- $\mathcal{H}_k^p$, the species which associated to any finite set V the set of large k-chains of rooted hypertrees on V.

Theorem (BO, 2013)

The species $\mathcal{H}_k^0$ and $\mathcal{H}_k^p$ satisfy:

$$\mathcal{H}_k^p = X \times \left(\mathcal{H}_k^0 + 1\right)$$
$$\mathcal{H}_k^0 = \mathbb{E}^+ \circ \mathcal{H}_{k-1}^0 \circ \mathcal{H}_k^p$$

Proof

Dimension of the homology

Proposition

The generating series of the species $\mathcal{H}_k^0$ satisfies:

$$\mathcal{C}_k^0 = \exp\left(\mathcal{C}_{k-1}^0 \circ \left(x\left(\mathcal{C}_k^0 + 1\right)\right)\right) - 1,$$
$$\mathcal{C}_1^0 = \exp\left(\exp\left(x\left(\mathcal{C}_1^0 + 1\right)\right) - 1\right) - 1$$

We thus get:

Theorem (adapted from [McCammond-Meier, 04])

The dimension of the unique homology group of the augmented hypertree poset is n^{n-1}.

Question:

Is it also the same action of the symmetric group ?

Action of the symmetric group on this poset

Outline

Definition

The cycle index series of a species F of a species F is the formal power series in an infinite number of variables $\mathfrak{p} = (p_1, p_2, p_3, \ldots)$ defined by:

$$Z_F(\mathfrak{p}) = \sum_{n \geqslant 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \ldots \right),$$

- with F^σ, is the set of F-structures fixed under the action of σ,
- and σ_i, the number of cycles of length i in the decomposition of σ into disjoint cycles.

Examples

- The cycle index series of the species of lists is $Z_{\mathbb{L}} = \frac{1}{1-p_1}$.
- The cycle index series of the species of non empty sets is $Z_{\mathbb{E}^+} = \exp(\sum_{k \geqslant 1} \frac{p_k}{k}) - 1$.

Operations

Operations on species give operations on their cycle index series:

Proposition

Let F and G be two species. Their cycle index series satisfy:

$$Z_{F+G} = Z_F + Z_G, \quad Z_{F \times G} = Z_F \times Z_G, \quad Z_{F \circ G} = Z_F \circ Z_G.$$

Definition

The suspension Σ_t of a cycle index series $f(p_1, p_2, p_3, \ldots)$ is defined by:

$$\Sigma f = -f(-p_1, -p_2, -p_3, \ldots).$$

Theorem (BO, 13; conjecture of [Chapoton, 05])

The cycle index series Z^0_{-1}, which gives the character for the action of $\mathfrak{S}_n$ on $\tilde{H}_{n-2}$, is given by:

$$\begin{aligned} Z^0_{-1} &= \Sigma\mathbb{C} \\ &= \Sigma\,\mathrm{PreLie}\,. \end{aligned}$$

Let us draw the parallel with partition posets !

Theorem (Stanley 82, Hanlon 81, Joyal 85)

The cycle index series which gives the character for the action of $\mathfrak{S}_n$ on $\tilde{H}_{n-1}(\Pi_n)$, is given by $\Sigma\,\mathrm{Lie}$.

Link with operads [Fresse 04] !

What is an operad ?

An operad is a vector species $\mathcal{F}$ (i.e. species with values in finite vector spaces) endowed with an associative product

$$\pi : \mathcal{F} \circ \mathcal{F} \to \mathcal{F}$$

and a unit.

Why do we care about operads ?

To any (algebraic) operad can be associated a type of algebras.
Proving properties on operads (such as Koszulness) help proving them for algebras.

Some Lie operads

Operads map a set V to a quotient of a vector space spanned by planar trees whose internal nodes are decorated by products and whose leaves are labelled by V. The composition is given by the grafting on the leaves.

- Lie generated by a binary product $[.;.]$ with relations Jacobi+anti-symmetry
- pre-Lie generated by a binary product $\curvearrowleft$ with the pre-Lie relation

$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y)$$

(basis given by Cayley trees)

- post-Lie generated by two binary products $[.;.]$ and $\lhd$ such that $[.;.]$ Lie bracket and

$$(x \lhd y) \lhd z - x \lhd (y \lhd z) - (x \lhd z) \lhd y + x \lhd (z \lhd y) = 0$$

$$[x, y] \lhd z = [x \lhd z, y] + [x, y \lhd z]$$

(basis given by Lie brackets of planar binary trees)

Partition posets [Fresse 04]

Minimal building set $=$ partition with only one part of size > 2

Nested sets of the poset Π_k: $\mathcal{NS}(\Pi_k)$= set of trees with leaves decorated by a partition of $\{1,\ldots,k\}$

Lemma

Given a partition $\pi = \{\pi_1,\ldots,\pi_k\}$,

$$[\hat{0},\pi] \simeq \Pi_k, \qquad [\pi,\hat{1}] \simeq \prod_{i=1}^{k} \Pi_{|\pi_i|}$$

Get an operad by grafting !

Back to the hypertree posets

We consider the poset HT_n (and not $\widehat{HT}_n$).

Lemma (Dupont-BDO 22+)

The Möbius number of HT_n is given by $\frac{(2n-1)!}{n!} = \sum_{\tau \in C(n)} \prod_{v \in V(\tau)} (deg(v) - 1)!$.

A006963 Number of planar embedded labeled trees with n nodes: (2n-3)!/(n-1)! for n >= 2, a(1) = 1. (Formerly M3076) 23

1, 1, 3, 20, 210, 3024, 55440, 1235520, 32432400, 980179200, 33522128640, 1279935820800, 53970627110400, 2490952020480000, 124903451312640000, 6761440164390912000, 393008709555221760000, 24412776311194951680000, 1613955767240110694400000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS For n>1: central terms of the triangle in A173333; cf. A001761, A001813. - Reinhard Zumkeller, Feb 19 2010

Can be obtained from the Vandermonde permanent of the first n positive integers; see A093883. - Clark Kimberling, Jan 02 2012

All trees can be embedded in the plane, but "planar embedded" means that orientation matters but rotation doesn't. For example, the n-star with n-1 edges has n! ways to label it, but rotation removes a factor of n-1. Another example, the n-path has n! ways to label it, but rotation removes a factor of 2. - Michael Somos, Aug 19 2014

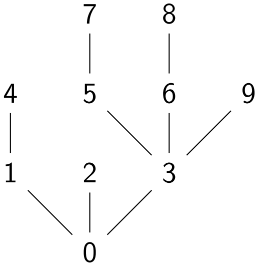
REFERENCES N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

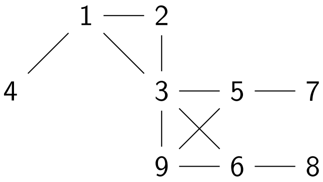
Hypertree posets and post-Lie operad

Minimal building set : Hypertrees with only one edge of cardinality > 2

Nested sets : pairs (T, N) where T is a tree and N is a tubing of its adjacency graph or equivalently doubled trees

$T =$





Composition of doubled trees

Theorem (Dupont-BDO,22+)

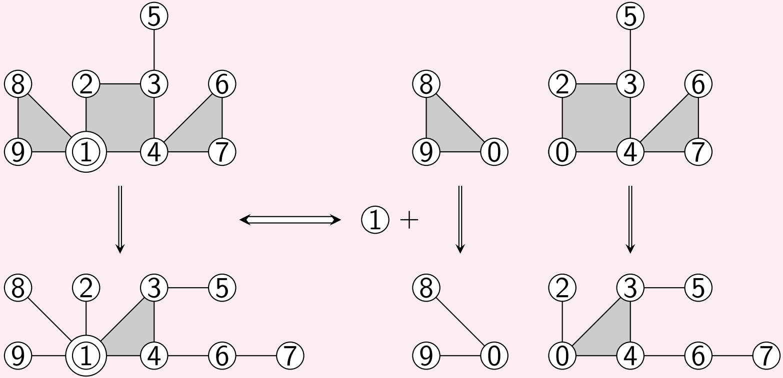
It is possible to define a (dg-) operad on the nested sets of the hypertree posets and it induces on the homology an operad which is (the suspension of the) post-Lie operad.

Thank you for your attention !

Proof of the first equation.

We separate the root and every edge containing it, putting $\{0\}$ where the root was,

$$\mathcal{H}_k^p = X \times \mathcal{H}_k^0 + X,$$



and of the second one!

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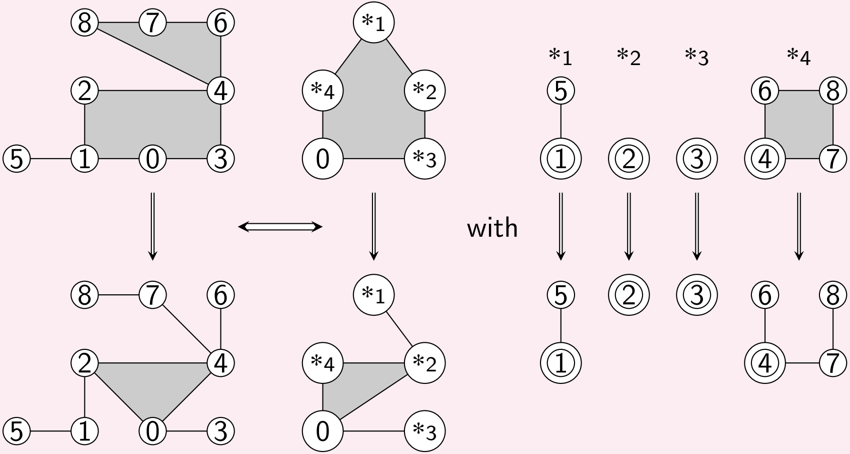
$$\mathcal{H}_k^0 = \mathbb{E}^+ \circ \mathcal{H}_{k-1}^0 \circ \mathcal{H}_k^p$$



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