Tridendriform structures on faces of hypergraph associahedra

Bérénice Delcroix-Oger
joint work with Jovana Obradović (Serbian Academy of Science) and
Pierre-Louis Curien (CNRS-IRIF, Université Paris Cité)

Université Paris Cité

SLC 87
Saint-Paul-en-Jarez, April 2022

Outline

1. Combinatorial shuffle
2. Hypergraph associahedra (a.k.a. nestoedra)
3. Splitting the shuffle product on faces of hypergraph associahedra

Combinatorial shuffle

Outline

Shuffle product on packed words [Chapoton, 00; Hivert-Novelli-Thibon]

A surjection $f : \{1, \ldots, n\} \to \{1, \ldots, d\}$ $(n \geqslant d)$ can be represented as a word $f(1) \ldots f(n)$ called packed word of length n, using all letters in $\{1, \ldots, d\}$.

Shuffle product on packed words [Chapoton, 00; Hivert-Novelli-Thibon]

A surjection $f : \{1, \ldots, n\} \to \{1, \ldots, d\}$ $(n \geqslant d)$ can be represented as a word $f(1) \ldots f(n)$ called packed word of length n, using all letters in $\{1, \ldots, d\}$.

To any map $g : \{1, \ldots, n\} \to \{i_1 < \ldots < i_k\}$ can be associated a set composition $SC_g = (g^{-1}(i_1), \ldots, g^{-1}(i_k))$. There is a unique surjection $pack(g) : \{1, \ldots, n\} \to \{1, \ldots, k\}$ having the same set composition as g.

Shuffle product on packed words [Chapoton, 00; Hivert-Novelli-Thibon]

A surjection $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, d\}$ $(n \geqslant d)$ can be represented as a word $f(1) \ldots f(n)$ called packed word of length n, using all letters in $\{1, \ldots, d\}$.

To any map $g : \{1, \ldots, n\} \rightarrow \{i_1 < \ldots < i_k\}$ can be associated a set composition $SC_g = (g^{-1}(i_1), \ldots, g^{-1}(i_k))$. There is a unique surjection $pack(g)$:$\{1, \ldots, n\} \rightarrow \{1, \ldots, k\}$ having the same set composition as g.

Example

$pack(154422) = 143322$

Shuffle product on packed words [Chapoton, 00; Hivert-Novelli-Thibon]

Definition

The vector space spanned by packed words can be endowed with a shuffle product defined by:

$$u * v = \sum a.b,$$

where the sum runs over all words a and b such that $pack(a) = u$, $pack(b) = v$ and the concatenation $a.b$ is a packed word.

Shuffle product on packed words [Chapoton, 00; Hivert-Novelli-Thibon]

Definition

The vector space spanned by packed words can be endowed with a shuffle product defined by:

$$u * v = \sum a.b,$$

where the sum runs over all words a and b such that $pack(a) = u$, $pack(b) = v$ and the concatenation $a.b$ is a packed word.

Examples:

$1 * 1 = 11 + 12 + 21$

$12 * 11 = 1211 + 1322 + 1233 + 2311$

Shuffle product on planar trees [Loday-Ronco, 04]

A planar tree is a combinatorial structure defined recursively by :

- $|$ is a PT
- $\vee(F_1, \ldots, F_n)$ is a PBT, if $F_1, \ldots, F_n$ are PBTs, for any $n \geqslant 2$.

Shuffle product on planar trees [Loday-Ronco, 04]

A planar tree is a combinatorial structure defined recursively by :

- $|$ is a PT
- $\vee(F_1, \dots, F_n)$ is a PBT, if $F_1, \dots, F_n$ are PBTs, for any $n \geqslant 2$.

Definition

The vector space spanned by PBT can be endowed with a shuffle product defined by:

$$| * T = T * \vee = T,$$

and for $T = \vee(T_1, \dots, T_k)$ and $S = \vee(S_1, \dots, S_p)$,

$$T * S = \vee(T * S_1, \dots, S_p) + \vee(T_1, \dots, T_k * S_1, \dots, S_p) + \vee(T_1, \dots, T_k * S)$$

Shuffle product on planar trees [Loday-Ronco, 04]

A planar tree is a combinatorial structure defined recursively by :

- $|$ is a PT
- $\vee(F_1, \dots, F_n)$ is a PBT, if $F_1, \dots, F_n$ are PBTs, for any $n \geqslant 2$.

Definition

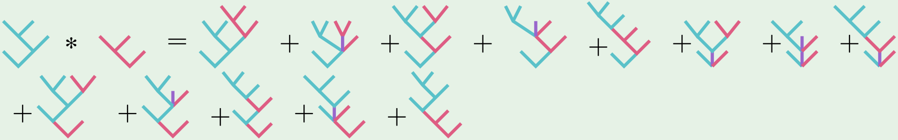
The vector space spanned by PBT can be endowed with a shuffle product defined by:

$$| * T = T * \vee = T,$$

and for $T = \vee(T_1, \dots, T_k)$ and $S = \vee(S_1, \dots, S_p)$,

$$T * S = \vee(T * S_1, \dots, S_p) + \vee(T_1, \dots, T_k * S_1, \dots, S_p) + \vee(T_1, \dots, T_k * S)$$

Example:



Main questions

- How to generate these combinatorial objects ?
- Are the algebras free ? What are their basis ?

Some shuffle algebras

	Packed words	PT
Free ?	yes [NT06 with Foissy07]	yes [LR04]
Basis	unsecable words	Infinitely many

Some shuffle algebras

	Packed words	PT
Free ?	yes [NT06 with Foissy07]	yes [LR04]
Basis	unsecable words	Infinitely many

Goal :

Find a smaller basis !

Idea:

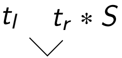
Three kinds of trees (looking at the root) : why not splitting in three the product $*$?

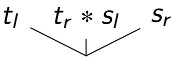
Inductive definition of tridendriform products on trees

If $T =$ 

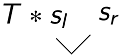
and $S =$ 

,

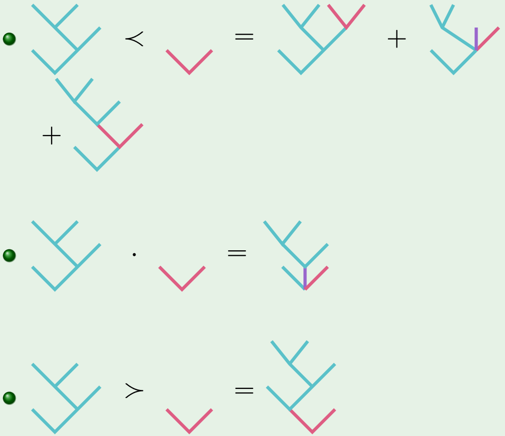
$T \prec S =$ 

$T \cdot S =$ 

and

$T \succ S =$ 

Examples :



Tridendriform algebras

Definition (Loday, Ronco, 2004 ; Chapoton 2002)

A tridendriform algebra is a vector space A endowed with products $\prec: A \otimes A \to A$, $\cdot : A \otimes A \to A$ and $\succ: A \otimes A \to A$, such that:

1. $(a \prec b) \prec c = a \prec (b * c)$,
2. $(a * b) \succ c = a \succ (b \succ c)$,
3. $(a \succ b) \prec c = a \succ (b \prec c)$,
4. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
5. $(a \succ b) \cdot c = a \succ (b \cdot c)$,
6. $(a \prec b) \cdot c = a \cdot (b \succ c)$,
7. $(a \cdot b) \prec c = a \cdot (b \prec c)$,

with $* = \prec + \cdot + \succ$

Algebra on packed words WQSym [Novelli-Thibon, 2006]

$$u \# v = \sum_{\substack{\text{pack}(\alpha)=u \\ \text{pack}(\beta)=v \\ c_\#}} \alpha\beta,$$

where $c_\# = \min(\alpha) < \min(\beta)$ for $\# = \prec$,
$c_\# = \min(\alpha) = \min(\beta)$ for $\# = \cdot$,
and $c_\# = \min(\alpha) > \min(\beta)$ for $\# = \succ$.

Example :

$$11 \succ 221 = 22221 + 33221 + 22331$$

$$11 \cdot 221 = 11221$$

$$11 \prec 221 = 11332$$

Algebra on packed words WQSym [Novelli-Thibon, 2006]

$$u \# v = \sum_{\substack{\text{pack}(\alpha)=u \\ \text{pack}(\beta)=v \\ c_\#}} \alpha\beta,$$

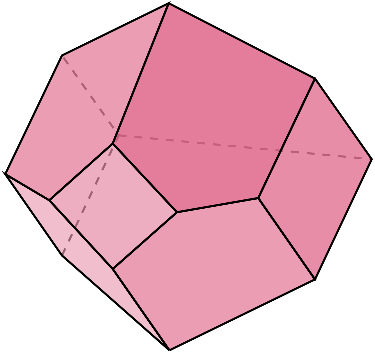
where $c_\# = \min(\alpha) < \min(\beta)$ for $\# = \prec$,
$c_\# = \min(\alpha) = \min(\beta)$ for $\# = \cdot$,
and $c_\# = \min(\alpha) > \min(\beta)$ for $\# = \succ$.

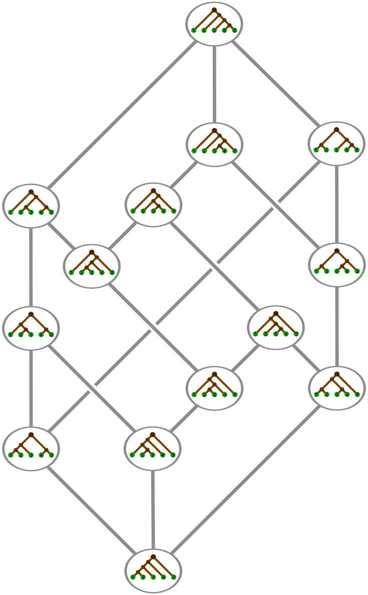
Example :

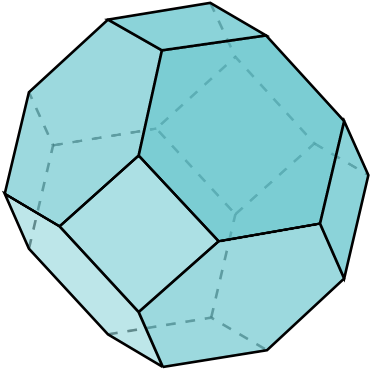
$$11 \succ 221 = 22221 + 33221 + 22331$$
$$11 \cdot 221 = 11221$$
$$11 \prec 221 = 11332$$

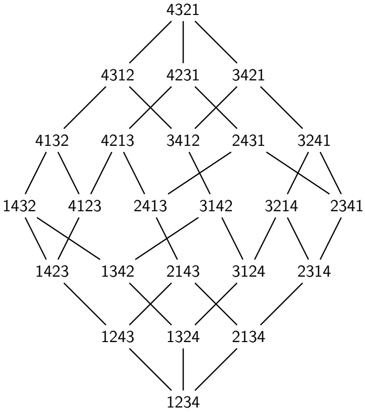
Tridendriform products $\Rightarrow$ WQSym free tridendriform algebra on infinitely many generators [Vong, Burgunder-Curien-Ronco, 2015]

Link with associahedra and permutohedra





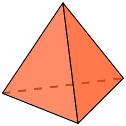



4321
4312 4231 3421
4132 4213 3412 2431 3241
1432 4123 2413 3142 3214 2341
1423 1342 2143 3124 2314
1243 1324 2134
1234

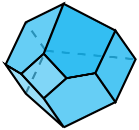
Hypergraph associahedra (a.k.a. nestoedra)

Outline

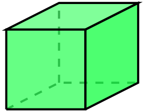
1. Combinatorial shuffle
2. Hypergraph associahedra (a.k.a. nestoedra)
3. Splitting the shuffle product on faces of hypergraph associahedra



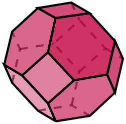
Simplices



Associahedra



Hypercubes



Permutohedra

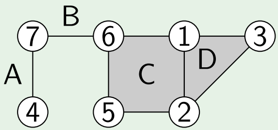
Hypergraphs

Definition

A hypergraph (on vertex set V) is a pair (V, E) where:

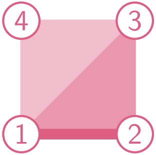
- V is a finite set, (the vertex set)
- E is a set of sets of size at least 2, $E \subset \mathcal{P}(V)$.

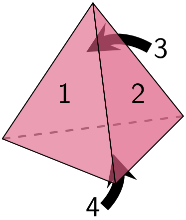
Example of an hypergraph on $[1;7]$

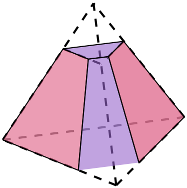


.

Hypergraph polytope [Došen, Petrić] (=nestohedra [Postnikov])







Constructs [Postnikov; Curien-Ivanovic-Obradović]

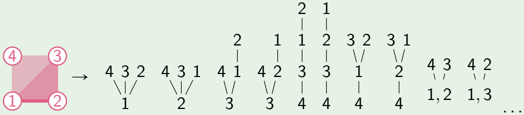
Constructs

A construct of a hypergraph H is defined inductively. For $E \subset V(H)$ (the set of vertices of H),

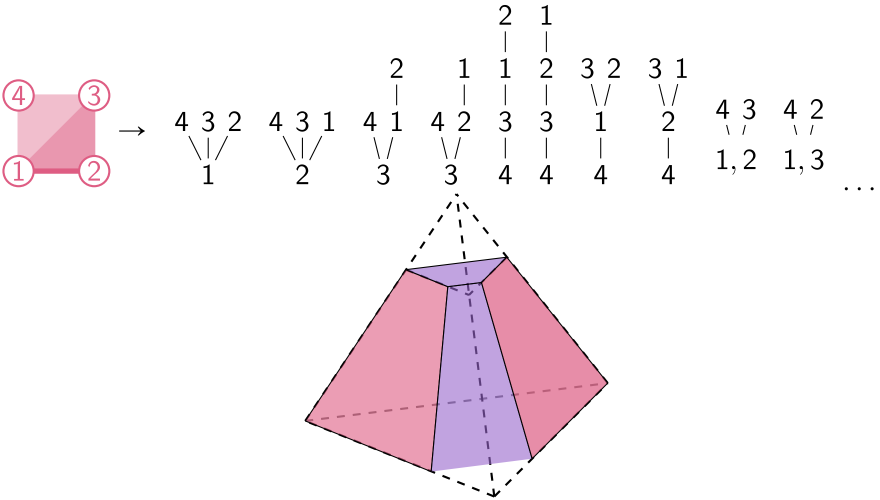
- If $E = V(H)$, the construct is the rooted tree with only one node labelled by E,
- Otherwise, denoting by $(T_1, \ldots, T_n)$ constructs on every connected component in $H - E$, a construct of H can be obtained by grafting these trees on a node labelled by E.

The set of constructs of a given hypergraph labels faces of the associated polytope.

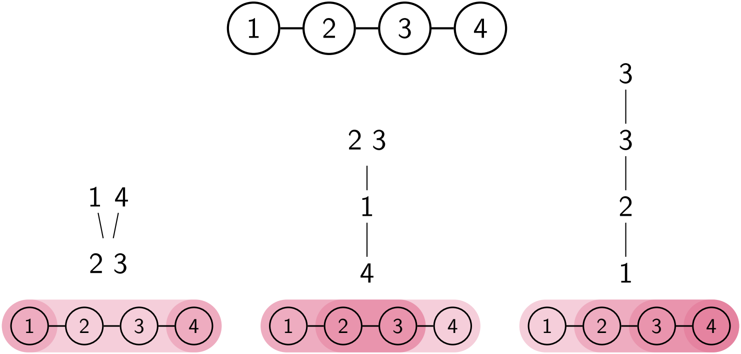
First example:



First example geometrically



Correspondence Tubings = Constructs = Spines



Splitting the shuffle product on faces of hypergraph associahedra

Outline

Shuffle product on faces of graph associahedra

Consider an admissible family $(G_n^i)_{\substack{1\leqslant i\leqslant s_n\\ n\geqslant 1}}$, with a collection of associative maps $\alpha(n,m) : \{s_1,\ldots,s_n\}\times\{s_1,\ldots,s_m\}\to\{s_1,\ldots,s_{n+m}\}$ such that $G_{n+m}^{\alpha(n,m)(i,j)}|_{\{1,\ldots,n\}} = G_n^i$ and $G_{n+m}^{\alpha(n,m)(i,j)}|_{\{n+1,\ldots,n+m\}} = G_m^j$ (up to a shift).

Shuffle product on faces of graph associahedra

Consider an admissible family $(G_n^i)_{\substack{1 \leqslant i \leqslant s_n \\ n \geqslant 1}}$, with a collection of associative maps $\alpha(n,m) : \{s_1, \ldots, s_n\} \times \{s_1, \ldots, s_m\} \to \{s_1, \ldots, s_{n+m}\}$ such that $G_{n+m}^{\alpha(n,m)(i,j)}|_{\{1,\ldots,n\}} = G_n^i$ and $G_{n+m}^{\alpha(n,m)(i,j)}|_{\{n+1,\ldots,n+m\}} = G_m^j$ (up to a shift).

Definition

Define on $T \in \mathrm{Cons}(G_n)$ and $W \in \mathrm{Cons}(G_m)$ the following product:

$$T * W = \sum U,$$

where the sum runs over all constructs U of G_{n+m} such that T (resp. W) is obtained from $U|_{\{1,\ldots,n\}}$ (resp. $U|_{\{n+1,\ldots,n+m\}}$) by merging some edges (resp. and shifting the labelling).

Shuffle product on faces of graph associahedra

Consider an admissible family $(G_n^i)_{\substack{1 \leqslant i \leqslant s_n \\ n \geqslant 1}}$, with a collection of associative maps $\alpha(n,m) : \{s_1, \ldots, s_n\} \times \{s_1, \ldots, s_m\} \to \{s_1, \ldots, s_{n+m}\}$ such that $G_{n+m}^{\alpha(n,m)(i,j)}|_{\{1,\ldots,n\}} = G_n^i$ and $G_{n+m}^{\alpha(n,m)(i,j)}|_{\{n+1,\ldots,n+m\}} = G_m^j$ (up to a shift).

Definition

Define on $T \in \mathrm{Cons}(G_n)$ and $W \in \mathrm{Cons}(G_m)$ the following product:

$$T * W = \sum U,$$

where the sum runs over all constructs U of G_{n+m} such that T (resp. W) is obtained from $U|_{\{1,\ldots,n\}}$ (resp. $U|_{\{n+1,\ldots,n+m\}}$) by merging some edges (resp. and shifting the labelling).

Theorem (Ronco, 12)

This product is associative.

Shuffle product on faces of graph associahedra

Consider an admissible family $(G_n^i)_{\substack{1\leqslant i\leqslant s_n\\ n\geqslant 1}}$, with a collection of associative maps $\alpha(n,m) : \{s_1,\ldots,s_n\}\times\{s_1,\ldots,s_m\}\to\{s_1,\ldots,s_{n+m}\}$ such that $G_{n+m}^{\alpha(n,m)(i,j)}|_{\{1,\ldots,n\}} = G_n^i$ and $G_{n+m}^{\alpha(n,m)(i,j)}|_{\{n+1,\ldots,n+m\}} = G_m^j$ (up to a shift).

Definition

Define on $T\in \mathrm{Cons}(G_n)$ and $W\in \mathrm{Cons}(G_m)$ the following product:

$$T * W = \sum U,$$

where the sum runs over all constructs U of G_{n+m} such that T (resp. W) is obtained from $U|_{\{1,\ldots,n\}}$ (resp. $U|_{\{n+1,\ldots,n+m\}}$) by merging some edges (resp. and shifting the labelling).

Theorem (Ronco, 12)

This product is associative.

Two goals

- Split this product
- Extend to hypergraph associahedra

Heuristics for a tridendriform structure

Let $\mathbf{H}^{\mathcal{X}}$ be a family of hypergraph polytopes, indexed by some finite sets $\mathcal{X}$ (sets of vertices of the associated hypergraphs).
For $S = A(S_1, \ldots, S_m)$ and $T = B(T_1, \ldots, T_n)$ two constructs of $\mathbf{H}^{\mathcal{X}}$ and $\mathbf{H}^{\mathcal{Y}}$ respectively ($\mathcal{X}, \mathcal{Y}$ disjoint), we would like to define the following operations

- $S \prec T$ as a sum of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** A**,**
- $S \succ T$ as a sum of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** B**,**
- $S \cdot T$ as a sum of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** $A \cup B$**.**

Tridendriform products defined on faces of simplices [Loday-Ronco, Chapoton]

On simplices, we get the following (triass) products, denoting by $(\mathcal{X}, A)$ the multipointed set whose underlying set is $\mathcal{X}$ and whose set of pointed elements is A:

$$(\mathcal{X}, A) \prec (\mathcal{Y}, B) = (\mathcal{X} \cup \mathcal{Y}, A)$$
$$(\mathcal{X}, A) \succ (\mathcal{Y}, B) = (\mathcal{X} \cup \mathcal{Y}, B)$$
$$(\mathcal{X}, A) \cdot (\mathcal{Y}, B) = (\mathcal{X} \cup \mathcal{Y}, A \cup B)$$

Tridendriform products defined on faces of hypercubes

Applying this construction to hypercube gives :

$$u \prec v = u\,(-^{|v|})$$
$$u \succ (v_1 + v_2) = \begin{cases} (u \star v_1) + v_2 & (v_1 \neq \epsilon) \\ u + v_2 & (v_1 = \epsilon) \end{cases}$$
$$u \cdot (v_1 + v_2) = u\,(-^{|v_1|}) \bullet v_2$$

where each word begins by a $+$ and the $+$ denotes the rightmost occurence of $+$.

Question

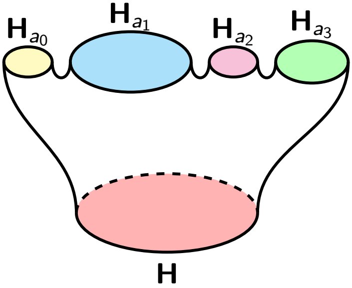
- How to formalize this construction ?
- How to deal with these examples which does not fit in the graph associahedra frame ? (lost edges, not associative)

Universe and preteam

The considered hypergraphs belong to a set of hypergraphs $\mathfrak{U}$, called universe.

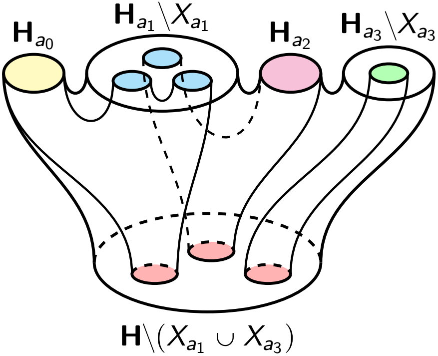
A preteam is a pair $\tau = (\{\mathbf{H}_a | a \in A\}, \mathbf{H})$ where

- $\{\mathbf{H}_a | a \in A, \mathbf{H}_a \in \mathfrak{U}\}$ is a set of pairwise disjoint hypergraphs, called participating hypergraphs
- $\mathbf{H} \in \mathfrak{U}$ is a hypergraph such that $H = \bigcup_{a \in A} H_a$, called supporting hypergraph.



Strict and semi-strict teams

A preteam is a (resp. semi-strict) strict team if the connected components obtained by deleting a subset X_a to every hypergraph $\mathbf{H}_a$ are in $\mathfrak{U}$ and included in the connected components of $\mathbf{H}\backslash\left(\bigcup_{a\in A} X_a\right)$ (resp. or totally disconnected)



$(X_{a_0} = X_{a_2} = \varnothing)$

Strict and semi-strict teams

A preteam is a (resp. semi-strict) strict team if the connected components obtained by deleting a subset X_a to every hypergraph $\mathbf{H}_a$ are in $\mathfrak{U}$ and included in the connected components of $\mathbf{H} \setminus (\bigcup_{a \in A} X_a)$ (resp. or totally disconnected)

Examples:

- Simplices
- Hypercubes
- Associahedra
- Permutohedra

Product

Considering a team E and denoting by δ a tuple of constructs of the team's participating hypergraphs, we inductively associate to δ a sum of constructs of the supporting hypergraph:

$$*(\delta) = \sum_{\emptyset \subset B \subseteq A} q^{|B|-1} \, (\bigcup_{b \in B} X_b)(*(\delta_1^B), \ldots, *(\delta_{n_B}^B)), \tag{1}$$

Polydendriform structure

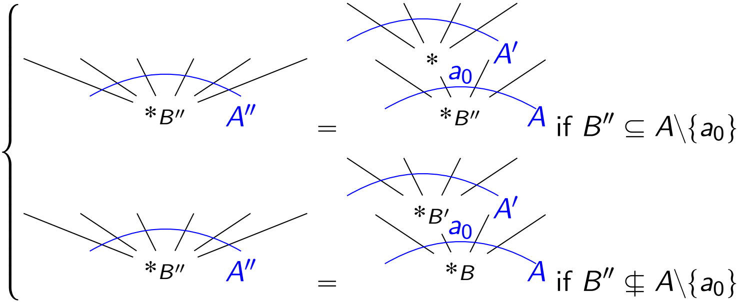
Let us introduce new operations

$$*_B(\delta) = (\bigcup_{b \in B} X_b)(*(\delta_1^B), \ldots, *(\delta_{n_B}^B))$$

such that the product splits

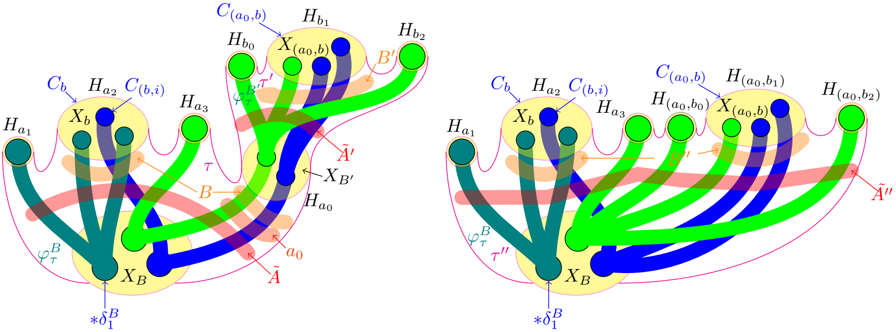
$$*(\delta) = \sum_{\emptyset \subset B \subseteq A} q^{|B|-1} *_B(\delta)$$

It satisfies relations:

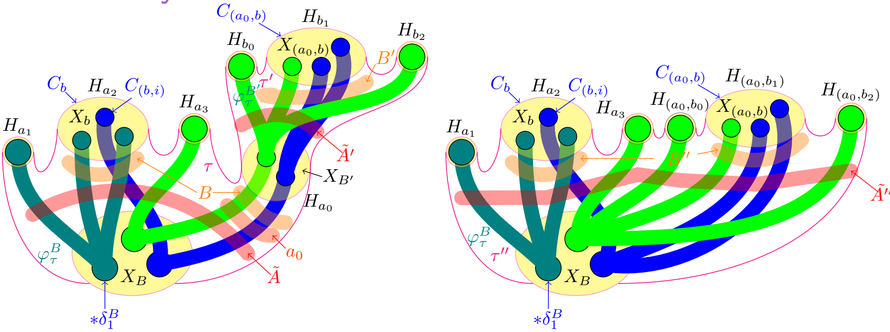


Associative clan

A set of (resp. semi-strict) strict team with "good" closure properties is called strict clan (each connected component obtained from the supporting hypergraph is itself a supporting hypergraph of a team).



Associativity of $*$



Theorem (Curien-D.O.-Obradović, 21+)

Consider a clan $\mathcal{C}$. The product $*$ is associative if

- *$\mathcal{C}$ is strict,*
- *or $\mathcal{C}$ is semi-strict and $q = -1$.*

- Strict clans: Associahedra, Permutohedra, Restrictohedra, ...
- Semi-strict clans: Simplices, Hypercubes, Cyclohedra, ...